BRST cohomology of the sum of two pure spinors

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Abstract

We study the zero mode cohomology of the sum of two pure spinors. The knowledge of this cohomology allows us to better understand the structure of the massless vertex operator of the Type IIB pure spinor superstring.

Contents

1 Introduction

Pure spinor formalism is perhaps the most promising approach to the high loop calculations in superstring theory. But in fact, it is even useful for the classical supergravity [1]. The pure spinor description of the Type IIB SUGRA is somewhat analogous to the pure spinor description of the supersymmetric Yang-Mills theory. But it is very different in details, and in fact much less understood.

In particular, we need to better understand the *massless* vertex operators. They are the cohomology classes describing the infinitesimal deformations of the given SUGRA solution.

In flat space-time, the massless vertex operators are *roughly speaking* products of the expression built on the left-movers, and the expression built from the right-movers. These left- and right-handed parts are very similar to the Maxwell vertices. This, however, does not work in curved space-time, where the separation into left- and right-movers does not exist. And even in flat space-time, there are subtleties at the near-zero momentum [2, 3].

A construction of the massless vertex in $AdS_5 \times S^5$ was suggested in [4]. Unfortunately it involved rather complicated calculations. In this paper we will get a better understanding of the construction, by setting up a slightly different cohomological perturbation scheme. The main component of this new scheme is the following "zero mode" BRST operator:

$$Q^{(0)} = (\lambda_L^\alpha + \lambda_R^\alpha)\,\frac{\partial}{\partial\theta^\alpha} \tag{1}$$

where λ_L^α and λ_R^α are both pure spinors, *i.e.* satisfy the constraints:

$$(\lambda_L \Gamma^m \lambda_L) = (\lambda_R \Gamma^m \lambda_R) = 0 \tag{2}$$

and θ^α are free Grassmann variables. It is assumed that $Q^{(0)}$ acts on *polynomials* of $\lambda_L^\alpha, \lambda_R^\alpha, \theta^\alpha$. This cohomology problem was first suggested in [5].

This $Q^{(0)}$ is the zeroth approximation to the BRST operator in $AdS_5 \times S^5$, acting on the chiral state. It turns out that the structure of $H(Q^{(0)})$ is rather rich. The full BRST operator induces on $H(Q^{(0)})$ the nilpotent operator d_1, then on the cohomologies of d_1 we get d_2, *etc.* The resulting spectral sequence converges to the Type IIB vertex operators. To understand the construction of vertex operators, the following facts about the cohomology of $Q^{(0)}$ are useful:

1. There is a class $\simeq \lambda^2 \theta^2$; the vertex is built by multiplying this class by some function $f(x)$
2. There is a class of the order $\lambda^3 \theta^3$, but its quantum numbers do not match the quantum numbers of the potential obstacle; therefore d_1 annihilates the vertex
3. However, there is a nontrivial class of the order $\lambda^3 \theta^5$. This implies that d_2 of the vertex is potentially nonzero. In fact vanishing of d_2 requires that $f(x)$ is a harmonic function. This is the expected on-shell condition.

Another construction of the massless vertex, which emphasizes the boundary-to-bulk structure, was suggested in [5, 6]. For that construction, the structure of the cohomology of $Q^{(0)}$ is also important.

2 Vertices from parabolic induction

In this section we will review the construction of pure spinor vertex operators in $AdS_5 \times S^5$, and explain how the knowledge of the cohomology of $Q^{(0)}$ allows to better organize the calculations.

2.1 Lie superalgebra $psl(4|4)$

Structure of the superalgebra The even subalgebra of $sl(4|4)$ is a direct sum of two Lie algebras[1]:

$$\mathbf{g}_{even} = \mathbf{g}_{\mathbf{up}} \oplus \mathbf{g}_{\mathbf{dn}} \tag{3}$$

Schematically, in the 4×4-block notations:

$$\mathbf{g} = \begin{bmatrix} \mathbf{g}_{\mathbf{up}} & \mathbf{n}_+ \\ \mathbf{n}_- & \mathbf{g}_{\mathbf{dn}} \end{bmatrix} \tag{4}$$

Notice that $\mathbf{n}_+$ and $\mathbf{n}_-$ are both odd abelian subalgebras $\mathbf{C}^{0|16}$.

Super coset space $AdS_5 \times S^5$ There is a "denominator" subalgebra:

$$\mathbf{g}_{\bar{0}} = sp(2) \oplus sp(2) \; \subset \; \mathbf{g}_{\text{even}} \tag{5}$$

The AdS space is the coset space [7]:

$$AdS_5 \times S^5 = \frac{PSL(4|4)}{Sp(2) \times Sp(2)} \tag{6}$$

The subalgebra $\mathbf{g}_{\bar{0}}$ is the stabilizer of a point in $AdS_5 \times S^5$. We will denote u^α the elements of the fundamental representation of $\mathbf{g}_{\text{up}} = sl(4)$ and v_a the elements of the anti-fundamental of $\mathbf{g}_{\text{dn}} = sl(4)$. Being a direct sum of two symplectic algebras, $\mathbf{g}_{\bar{0}}$ can be characterized as a stabilizer of a pair of symplectic forms $\omega^{\alpha\beta}$ and ω_{ab} [7]. (With a slight abuse of notations, we use the same letter ω for both of them.) We introduce some additional notations, see also Section 3.2.1:

$$u \cup v = u^\alpha \omega_{\alpha\beta} v^\beta \; , \quad u \cap v = u_a \omega^{ab} v_b \; , \quad ||u|| = u^{\alpha\beta}\omega_{\beta\alpha} \texttt{ or } u_{ab}\omega^{ba} \tag{7}$$

2.2 Parabolic induction

The idea of [4] is to first construct the vertex for the "chiral" states — those states which are annihilated by all the elements of $\mathbf{n}_-$. Then, having the vertex for the chiral states, we can obtain the general vertex by applying the

[1]The real form used in AdS/CFT is $\mathbf{g}_{\mathbf{up}} = \mathbf{su(2,2)}$ and $\mathbf{g}_{\mathbf{dn}} = \mathbf{su(4)}$.

$PSL(4|4)$ rotations. Let us introduce the following coordinates on the group manifold $PSL(4|4)$:

$$g = e^{\omega} e^{\theta_+} e^{x} e^{\theta_-} \tag{8}$$

where $\omega \in \mathbf{g}_{\bar{0}}$, $\theta_\pm \in \mathbf{n}_\pm$, and x is in the complement of ω in $\mathbf{g}_{\text{even}} = \mathbf{g}_{\bar{2}} + \mathbf{g}_{\bar{0}}$.

For the chiral state, the vertex:

- will not depend on θ_-
- will transform in a given representation L of $\mathbf{g}_{\text{even}}$ under the shifts of x

We therefore use the *parabolic induction* from the following parabolic subalgebra:

$$\mathbf{p} = \left[\begin{array}{cc} \mathbf{g}_{\mathbf{up}} & 0 \\ \mathbf{n}_- & \mathbf{g}_{\mathbf{dn}} \end{array} \right] \tag{9}$$

We start with a representation L of the *bosonic* Lie algebra $\mathbf{g}_{\text{even}} = \mathbf{g}_{\mathbf{up}} \oplus \mathbf{g}_{\mathbf{dn}}$. We can only apply our construction in the case when L satisfies the following properties:

1. The quadratic Casimir of $\mathbf{g}_{\text{even}}$ vanishes on L
2. In a dual representation[2] L', exists a vector $\Omega \in L'$ such that the subalgebra $\mathbf{g}_{\bar{0}} \subset \mathbf{g}_{\text{even}}$ annihilates Ω.

Kac module. Chiral and non-chiral elements. We will construct some vertex for every element of the Kac module:

$$\text{Ind}_{\mathbf{p}}^{\mathbf{g}} L = \text{U}\mathbf{g} \otimes_{\mathbf{p}} L \tag{10}$$

Definition of chiral *vs.* **non-chiral** Those elements of $\text{Ind}_{\mathbf{p}}^{\mathbf{g}} L$ which are of the type $\mathbf{1} \otimes l$, where $\mathbf{1}$ is the unit of $\text{U}\mathbf{g}$ and l is a vector in L, will be called *chiral* elements. All other elements will be called *non-chiral*.

It was argued in [4] that most of elements of $\text{Ind}_{\mathbf{p}}^{\mathbf{g}} L$ give by our construction BRST-trivial vertices[3]. But all chiral and some non-chiral elements give BRST-nontrivial vertices.

[2] For any representation L, the *dual* representation L' is on the space of all linear functions on L

[3] this follows from the consistency of AdS/CFT, and also from identifying the R-charge of the Type IIB SUGRA; chiral states have R-charge $+2$, lowering operators decrease the R-charge, the R-charge cannot be less than -2

Definition of $v(\theta_+, \lambda)$ In order to construct the vertex, we first define $v(\theta_+, \lambda) \in L'$ — a function of θ_+ and λ taking values in L'. It is defined in terms of Ω:

$$v(\theta_+, \lambda) = \; ||\lambda_{R+} \cup \theta_+ \cap \lambda_{L+} \cup \theta_+|| \, \Omega + \dots \tag{11}$$

where ... stand for the terms of the higher order in θ_+ which we will define later.

Vertex for chiral states. The ansatz for the massless vertex corresponding to the *chiral* state $\mathbf{1} \otimes l$ is, using the coordinates defined in (8):

$$V(\theta_+, x, \lambda) \; = \langle v(\theta_+, \lambda), e^x l \rangle \tag{12}$$

Vertex for non-chiral states. For a general (*i.e.* non-chiral) state:

$$(\eta_{1+} \cdots \eta_{k+}) \otimes l$$

the vertex $V(\theta_+, \theta_-, x, \lambda)$ can be calculated in the following way. Remembering that $g = e^\omega e^{\theta_+} e^x e^{\theta_-}$, we write:

$$\begin{aligned} V(\theta_+, \theta_-, x, \lambda) &= \langle v(\theta_+, \lambda), \;\; g \quad (\eta_{1+} \cdots \eta_{k+}) \otimes l \rangle \;\; = \\ &= \langle v(\theta_+, \lambda), \; e^\omega e^{\theta_+} e^x e^{\theta_-} \; (\eta_{1+} \cdots \eta_{k+}) \otimes l \rangle \end{aligned} \tag{13}$$

then expand in θ_- and pass all the θ_- through η_+'s until they all are eaten into the commutators with η_+ resulting in elements of $\mathbf{g}_{\text{even}}$, which then rotate l as $l \in L$ — a representation of $\mathbf{g}_{\text{even}}$. In other words:

- vertex operators for non-chiral states are obtained from the vertex operators for the chiral state (12) by applying supersymmetry transformations

The action of the BRST operator in these notations is, schematically:

$$Q = Q_L + Q_R \tag{14}$$

$$\epsilon Q_L v = \; (\epsilon\lambda_{L+} + (\theta_+ \cap \epsilon\lambda_{L+} \cup \theta_+)) \, \frac{\partial}{\partial\theta_+} \; v \; + \; (\epsilon\lambda_{L+}\Gamma^m\theta_+) t^2_m v \tag{15}$$

$$\epsilon Q_R v = \; (\epsilon\lambda_{R+} - (\theta_+ \cap \epsilon\lambda_{R+} \cup \theta_+)) \, \frac{\partial}{\partial\theta_+} \; v \; - \; (\epsilon\lambda_{R+}\Gamma^m\theta_+) t^2_m v \tag{16}$$

where t^2_m are generators of $\mathbf{g}_{\bar{2}}$. Details are in [4].

2.3 Perturbation theory

Let us develop a perturbation theory considering $\theta_+,\ \lambda_{L+},\ \lambda_{R+}$ as small of the same order ε. In other words, let us consider the expansion of v in powers of θ_+. The zeroth order approximation to Q is:

$$Q^{(0)} = (\lambda_{L+} + \lambda_{R+})\frac{\partial}{\partial\theta_+} \tag{17}$$

Therefore to understand the structure of the vertex we need to start with calculating the cohomology of (17). We observe the following facts.

Order $\lambda^2\theta^2$ The cohomology of $Q^{(0)}$ at the order $\lambda^2\theta^2$ is generated by the coefficient of Ω in (11):

$$\Phi^{[\lambda^2\theta^2]}_{\text{scalar}} = ||\lambda_{R+}\cup\theta_+\cap\lambda_{L+}\cup\theta_+|| \tag{18}$$

and some $\Phi^{[\lambda^2\theta^2]}_{2-\text{form}}$.

The second approximation $Q^{(1)}$ brings $||\lambda_{R+}\cup\theta_+\cap\lambda_{L+}\cup\theta_+||\ \Omega$ to something of the order $\lambda^3\theta^3$.

Order $\lambda^3\theta^3$ The cohomology of $Q^{(0)}$ at the order $\lambda^3\theta^3$ is nonzero. However, it does not create an obstacle to completing the terms of the order $\lambda^2\theta^4$ in $v(\theta_+,\lambda)$. This is because, even being nonzero, its quantum numbers (*i.e.* representation content under $\mathbf{g}_{\bar{0}}$) do not match the quantum numbers of a potential obstacle. Indeed, the question is whether we could cancel the last term in (15) and (16), *i.e.* the terms proportional to $t_m^2\Omega$. But Ω is $\mathbf{g}_{\bar{0}}$-invariant, therefore $t_m^2\Omega$ transforms as a vector of the tangent space to sphere, plus a vector of the tangent space to AdS. But looking at either (56), (57), (58) and (59), or at (34), we see that there is no class of the type $\lambda^3\theta^3$ with such quantum numbers.

Therefore there is no obstacle to completing the terms of the order $\lambda^2\theta^4$ in $v(\theta_+,\lambda)$.

Order $\lambda^3\theta^5$ But the action of $Q^{(1)}$ on these terms of the order $\lambda^2\theta^4$ produces terms of the order $\lambda^3\theta^5$. It turns out that there is precisely one cohomology class at the order $\lambda^3\theta^5$. But the contribution of this term to Qv is proportional to the action of the Casimir of $\mathbf{g}_{\text{even}}$ on Ω, which is zero by the assumption.

Order $\lambda\theta$ There are two nontrivial cohomology classes at the order $\lambda\theta$:

$$\Phi^{[\lambda\theta]}_{\rm up} = \lambda_{L+} \overset{0}{\cap} \theta_+ + \theta_+ \overset{0}{\cap} \lambda_{L+} - (\theta_+ \overset{0}{\cap} \lambda_{R+} + \lambda_{R+} \overset{0}{\cap} \theta_+) \tag{19}$$

$$\Phi^{[\lambda\theta]}_{\rm dn} = \lambda_{L+} \overset{0}{\cup} \theta_+ + \theta_+ \overset{0}{\cup} \lambda_{L+} - (\theta_+ \overset{0}{\cup} \lambda_{R+} + \lambda_{R+} \overset{0}{\cup} \theta_+) \tag{20}$$

Is it possible that $v = Qw$ where w is composed of $\Phi^{[\lambda\theta]}_{\rm up}$ and $\Phi^{[\lambda\theta]}_{\rm dn}$ plus terms of the higher order in θ? As explained in [4], this is not possible for the following reason. Consider the transformation E which exchanges $\lambda_{L+} \leftrightarrow \lambda_{R+}$ and multiplies θ_+ by i:

$$E\lambda_{L+} = \lambda_{R+}\ ,\ E\lambda_{R+} = \lambda_{L+}\ ,\ E\theta_+ = i\theta_+ \tag{21}$$

This is a symmetry of the BRST complex, in the following sense:

$$EQ = -iQE \tag{22}$$

We observe that $E\Phi^{[\lambda\theta]}_{\rm up} = -i\Phi^{[\lambda\theta]}_{\rm up}$ and $E\Phi^{[\lambda\theta]}_{\rm dn} = -i\Phi^{[\lambda\theta]}_{\rm dn}$, therefore $EQ\Phi^{[\lambda\theta]} = -Q\Phi^{[\lambda\theta]}$. At the same time:

$$E\ ||\lambda_{R+} \cup \theta_+ \cap \lambda_{L+} \cup \theta_+|| = ||\lambda_{R+} \cup \theta_+ \cap \lambda_{L+} \cup \theta_+|| \tag{23}$$

This means that v cannot be cancelled by $\Phi^{[\lambda\theta]}_{\rm up}$ and $\Phi^{[\lambda\theta]}_{\rm dn}$.

Is it possible that $v = Qw$ where w is not chiral (*i.e.* depends on both θ_+ and θ_-)? This depends on the properties of L. For the type of problems usually encountered in AdS/CFT correspondence L is a tensor product of some representation of $\mathbf{g}_{\rm up} = so(2,4)$ and a *finite-dimensional* representation of $\mathbf{g}_{\rm dn} = so(6)$. Suppose that the quantum numbers of this finite-dimensional representation of $so(6)$ are large enough. Then [8] implies that, when $v = Qw$ and v transforms in $\mathrm{Coind}^{\mathbf{g}}_{\mathbf{p}} L'$, then w can also be chosen to transform in the same representation $\mathrm{Coind}^{\mathbf{g}}_{\mathbf{p}} L'$. Therefore, if v represents a nontrivial cohomology class in the chiral BRST complex, then it also represents a nontrivial cohomology class in the full BRST complex.

3 Cohomology of the sum of two pure spinors.

In this section we will give the result for the cohomology of the nilpotent operator:

$$Q^{(0)} = (\lambda_{L+} + \lambda_{R+})\frac{\partial}{\partial\theta_+} \tag{24}$$

where λ_{L+} and λ_{R+} are two different pure spinors, acting on the polynomials of $\lambda_L, \lambda_R, \theta$, up to the ghost number 3. The derivation of this results will follow in the next sections.

3.1 Table of cohomology

Up to the ghost number three, we find the following cohomology classes. Smallcase latin letters enumerate the basis vecotrs in $\mathbf{C}^{10}$, and capital letters are spinor indices.

Ghost number 1

$$\Phi^{[\lambda]} = \lambda_L - \lambda_R \tag{25}$$

$$\left(\Phi^{[\lambda\theta]}\right)^p = ((\lambda_L - \lambda_R)\Gamma^p\theta) \tag{26}$$

Ghost number 2

$$\left(\Phi^{[\lambda^2]}\right)^{pqrst} = \left((\lambda_L - \lambda_R)\Gamma^{pqrst}_{AB}(\lambda_L - \lambda_R)\right) \tag{27}$$

$$\left(\Phi^{[\lambda^2\theta]}\right)^{pA} = ((\lambda_L - \lambda_R)\Gamma^p\theta)\ (\lambda_L^A - \lambda_R^A)\ \mathtt{mod}\ \ \ldots \tag{28}$$

$$\Phi^{[\lambda^2\theta^2]}_{\mathtt{scalar}} = (\lambda_L\Gamma^p\theta)(\lambda_R\Gamma^p\theta) \tag{29}$$

$$\left(\Phi^{[\lambda^2\theta^2]}_{\mathtt{2-form}}\right)_{pq} = (\theta\Gamma_{r_1r_2r_3}\theta)(\lambda_L\Gamma_{pqr_1r_2r_3}\lambda_R) - 18(\theta\Gamma_{pqr}\theta)(\lambda_L\Gamma^r\lambda_R) \tag{30}$$

Notice that all classes of the ghost number 2 except $\Phi^{[\lambda^2\theta^2]}_{\mathtt{scalar}}$ are even under $\lambda_L \leftrightarrow \lambda_R$; the class $\Phi^{[\lambda^2\theta^2]}_{\mathtt{scalar}}$ is odd under $\lambda_L \leftrightarrow \lambda_R$.

Ghost number 3

$$\left(\Phi^{[\lambda^3]}\right)^{ABC} = (\lambda_L^A - \lambda_R^A)\ (\lambda_L^B - \lambda_R^B)\ (\lambda_L^C - \lambda_R^C)\ \texttt{mod}\ \ \ldots \tag{31}$$

$$\left(\Phi^{[\lambda^3\theta]}\right)^{pAB} = ((\lambda_L - \lambda_R)\Gamma^p\theta)\ (\lambda_L^A - \lambda_R^A)\ (\lambda_L^B - \lambda_R^B)\ \texttt{mod}\ \ \ldots \tag{32}$$

$$\begin{aligned}\left(\Phi^{[\lambda^3\theta^2]}\right)^A_{pq} &= (\lambda_L^A - \lambda_R^A)\ \left(\Phi^{[\lambda^2\theta^2]}_{\texttt{2-form}}\right)_{[pq]} \\ &\qquad \texttt{mod}\ \ (\Gamma_{pq}\Gamma^{rs}(\lambda_L - \lambda_R))_A\ \left(\Phi^{[\lambda^2\theta^2]}_{\texttt{2-form}}\right)_{[rs]}\end{aligned} \tag{33}$$

$$\Phi^{[\lambda^3\theta^3]}_{pqr} = \Gamma^{AB}_{pqr}\frac{\partial}{\partial\theta^A}\frac{\partial}{\partial\theta^B}\Phi^{[\lambda^3\theta^5]} \tag{34}$$

$$\Phi^{[\lambda^3\theta^4]}_A = \frac{\partial}{\partial\theta^A}\Phi^{[\lambda^3\theta^5]} \tag{35}$$

$$\begin{aligned}\Phi^{[\lambda^3\theta^5]} = &\left(1 - \frac{5}{3}\left(\lambda_L\frac{\partial}{\partial\lambda_R}\right)\right)(\lambda_R\Gamma^p\theta)(\lambda_R\Gamma^q\theta)(\lambda_R\Gamma^r\theta)(\theta\Gamma_{pqr}\theta) - \\ &- \left(1 - \frac{5}{3}\left(\lambda_R\frac{\partial}{\partial\lambda_L}\right)\right)(\lambda_L\Gamma^p\theta)(\lambda_L\Gamma^q\theta)(\lambda_L\Gamma^r\theta)(\theta\Gamma_{pqr}\theta)\end{aligned} \tag{36}$$

In Eq. (36) the notation $\lambda_R\frac{\partial}{\partial\lambda_L}$ stands for a formal differentiation w.r.to λ, without taking into account the pure spinor constraint. In other words, it is just a substitution of *one* of the λ_L by λ_R. The notation $\lambda_R\frac{\partial}{\partial\lambda_L}$ has the same meaning.

All classes of the ghost number 3 are odd under $\lambda_L \leftrightarrow \lambda_R$.

Cohomology as a module over the algebra of pure spinors The cohomology classes form a module over the commutative algebra

$$\mathcal{R} = \mathbb{C}[\lambda_L^1, \ldots, \lambda_L^{16},\ \lambda_R^1, \ldots, \lambda_R^{16}]/\Big((\lambda_L\Gamma^m\lambda_L) = (\lambda_R\Gamma^m\lambda_R) = 0\Big)$$

It is generated by:

$$1, \Psi^{[\lambda\theta]}, \Psi^{[\lambda^2\theta^2]}_{scalar}, \Psi^{[\lambda^2\theta^2]}_{2-form}, \Psi^{[\lambda^3\theta^3]}, \Psi^{[\lambda^3\theta^4]}, \Psi^{[\lambda^3\theta^5]} \tag{37}$$

All other cohomology classes can be obtained from these generators by means of multiplications of polynormials of λ_L and λ_R. This module, however, is not free; its structure is discussed in Sections 3.3 and 4.3.2.

3.2 Result in AdS notations

3.2.1 Notations and abbreviations

Pure spinors as a cone over the group manifold. We find it technically useful to consider two different parametrizations of spinors. One is the usual picture of 10-dimensional Weyl spinors, which form a representation of the even part of the Clifford algebra of $so(10)$. The other is obtained by splitting $so(10) = so(5) \oplus so(5)$ and using the fact that $so(5) \simeq sp(2)$. From this point of view, each spinor has an upper and a lower index: θ^{α}_{a}. Both indices are in the fundamental representation of the corresponding $so(5)$, and $4 \times 4 = 16$. A *generic* pure spinor corresponds to a point of the cone over the group manifold $Sp(2)$; see Section 4.4.1.

Therefore there are two different schemes of notations, which we will call "AdS notations" and "flat space notations".

AdS	flat space
$\lambda^{[\alpha}_{a}\omega^{ab}\theta^{\beta]}_{b}$	$(\lambda\Gamma^{m}\theta)$ `for` $m \in \{0, \ldots 4\}$
$\lambda^{\alpha}_{[a}\omega_{\alpha\beta}\theta^{\beta}_{b]}$	$(\lambda\Gamma^{m}\theta)$ `for` $m \in \{5, \ldots 9\}$
Eq. (40)	$(\lambda\Gamma^{m}\theta)\Gamma_{m}\theta$

The difference between these two schemes is purely notational. There is no difference between the tangent space at the point of $AdS_5 \times S^5$ and the tangent space at the point of $\mathbf{R}^{10}$. In particlar, the zero mode BRST operator (24) is the same in AdS and in flat space.

Abbreviations In the following formulas, we will use following abbreviated notations. We will write $v^{\bullet}$ instead of v^{α}, just to indicate that there is an upper index, when we don't bother about the value of the index. For example, we write $\theta^{\bullet} \cap \lambda^{\bullet}$ (or sometimes $(\theta \cap \lambda)^{\bullet\bullet}$) instead of $\theta^{\alpha}_{a}\omega^{ab}\lambda^{\beta}_{b}$, just to indicate that there are two uncontracted indices (α and β). Similarly, $\theta_{\bullet} \cup \lambda_{\bullet}$ stands for $\theta^{\alpha}_{a}\omega_{\alpha\beta}\lambda^{\beta}_{b}$. We will sometimes omit indices altogether, and simply write $\theta \cup \lambda$. Furthermore, we will write:

$$\{u \cap v\} = u \cap v + v \cap u \tag{38}$$

$$u^{\bullet} \overset{0}{\cap} v^{\bullet} = u^{\bullet} \cap v^{\bullet} - \frac{1}{4}\omega^{\bullet\bullet} \; ||u \cap v|| \tag{39}$$

Notice that $||u \overset{0}{\cap} v|| = 0$ because ${}^{\bullet}\cap \cup_{\bullet} = 4\delta^{\bullet}_{\bullet}$.

We will also denote:

$$[\theta \cap \lambda \cup \theta]_\Gamma = \; 3\,\theta \cap \lambda \cup \theta + \left(\lambda \frac{\partial}{\partial \theta}\right) \theta \cap \theta \cup \theta \tag{40}$$

3.2.2 Table of cohomology

Cohomology of $\lambda_{L+}\frac{\partial}{\partial\theta_+}$ This was calculated in [9]; see also the Appendix of [10]. In our notations, the nontrivial classes are:

$$\begin{aligned}
&1,\\
&\lambda_{L+} \overset{0}{\cap} \theta_+ \;, \quad \lambda_{L+} \overset{0}{\cup} \theta_+ \;,\\
&\theta_+ \cap \lambda_{L+} \cup \theta_+ \;,\\
&\theta_+ \cap \lambda_{L+} \cup \theta_+ \cap (\lambda_{L+} \overset{0}{\cup} \theta_+) \;,\\
&\theta_+ \cup \lambda_{L+} \cap \theta_+ \cup \theta_+ \cap \lambda_{L+} \cup \theta_+ \;, \quad \theta_+ \cap \lambda_{L+} \cup \theta_+ \cap \theta_+ \cup \lambda_{L+} \cap \theta_+ \;,\\
&||\theta_+ \cap \lambda_{L+} \cup \theta_+ \cap \theta_+ \cup \lambda_{L+} \cap \theta_+ \cup \lambda_{L+} \overset{0}{\cap} \theta_+||
\end{aligned} \tag{41}$$

In Section 4.2 we will explain how to reduce the calculation of the cohomology of $Q^{(0)}$ to the cohomology of $\lambda_{L+}\frac{\partial}{\partial\theta_+}$ and $\lambda_{R+}\frac{\partial}{\partial\theta_+}$.

Cohomology of $Q^{(0)}$ In Sections 4 and 5 we will calculate the cohomology of $Q^{(0)}$ up to the ghost number three. Here we just give the result of the calculation. The cohomology of $Q^{(0)}$ up to the ghost number three is:

$$\Phi^{[1]} = 1 \tag{42}$$

$$\Phi^{[\lambda]} = (\lambda_{L+} - \lambda_{R+})^{\bullet}_{\bullet} \tag{43}$$

$$\Phi^{[\lambda\theta]}_{\rm up} = \{(\lambda_{L+} - \lambda_{R+}) \overset{0}{\cap} \theta_+\}^{\bullet\bullet} \tag{44}$$

$$\Phi^{[\lambda\theta]}_{\rm dn} = \{(\lambda_{L+} - \lambda_{R+}) \overset{0}{\cup} \theta_+\}_{\bullet\bullet} \tag{45}$$

$$\begin{aligned}
\Phi^{[\lambda^2]} = \; & (\lambda_{L+} - \lambda_{R+})^{\bullet}_{\bullet}\, (\lambda_{L+} - \lambda_{R+})^{\bullet}_{\bullet}\\
& \texttt{mod}\;\; \omega^{\bullet\bullet}((\lambda_{L+} - \lambda_{R+}) \cup (\lambda_{L+} - \lambda_{R+}))_{\bullet\bullet}\\
& \texttt{and}\;\; \omega_{\bullet\bullet}((\lambda_{L+} - \lambda_{R+}) \cap (\lambda_{L+} - \lambda_{R+}))^{\bullet\bullet}
\end{aligned} \tag{46}$$

$$\Phi^{[\lambda^2\theta]}_{\mathrm{up}} = \{(\lambda_{L+} - \lambda_{R+}) \overset{0}{\cap} \theta_+\}^{\bullet\bullet} \, (\lambda_{L+} - \lambda_{R+})^{\bullet}_{\bullet} \tag{47}$$
$$\texttt{mod} \ \left(\{(\lambda_{L+} - \lambda_{R+}) \overset{0}{\cap} \theta_+\} \cup (\lambda_{L+} - \lambda_{R+}) \right)^{\bullet}_{\bullet} \ \omega^{\bullet\bullet}$$
$$\Phi^{[\lambda^2\theta]}_{\mathrm{dn}} = \{(\lambda_{L+} - \lambda_{R+}) \overset{0}{\cup} \theta_+\}_{\bullet\bullet} \, (\lambda_{L+} - \lambda_{R+})^{\bullet}_{\bullet} \tag{48}$$
$$\texttt{mod} \ \left(\{(\lambda_{L+} - \lambda_{R+}) \overset{0}{\cup} \theta_+\} \cap (\lambda_{L+} - \lambda_{R+}) \right)^{\bullet}_{\bullet} \ \omega_{\bullet\bullet}$$

$$\Phi^{[\lambda^3]} = (\lambda_{L+} - \lambda_{R+})^{\bullet}_{\bullet} \ (\lambda_{L+} - \lambda_{R+})^{\bullet}_{\bullet} \ (\lambda_{L+} - \lambda_{R+})^{\bullet}_{\bullet} \tag{49}$$
$$\texttt{mod some equivalence relations}$$

$$\Phi^{[\lambda^2\theta^2]}_{\texttt{scalar}} = ||\lambda_{L+} \cap \theta_+ \cup \lambda_{R+} \cap \theta_+|| \tag{50}$$
$$\begin{aligned}\Phi^{[\lambda^2\theta^2]}_{\texttt{up}} = {} & \lambda^{(\bullet}_{L+} \cap \theta_+ \cup \theta_+ \cap \lambda^{\bullet)}_{R+} + \lambda^{(\bullet}_{L+} \cap \lambda_{R+} \cup \theta_+ \cap \theta^{\bullet)}_{+} + \\ & + \theta^{(\bullet}_{+} \cap \theta_+ \cup \lambda_{L+} \cap \lambda^{\bullet)}_{R+} + \lambda^{(\bullet}_{R+} \cap \theta_+ \cup \lambda_{L+} \cap \theta^{\bullet)}_{+} - \\ & - \theta^{(\bullet}_{+} \cap \lambda_{R+} \cup \lambda_{L+} \cap \theta^{\bullet)}_{+} + \theta^{(\bullet}_{+} \cap \lambda_{R+} \cup \theta_+ \cap \lambda^{\bullet)}_{L+}\end{aligned} \tag{51}$$
$$\begin{aligned}\Phi^{[\lambda^2\theta^2]}_{\texttt{dn}} = {} & \lambda_{L+(\bullet} \cup \theta_+ \cap \theta_+ \cup \lambda_{R+\bullet)} + \lambda_{L+(\bullet} \cup \lambda_{R+} \cap \theta_+ \cup \theta_{+\bullet)} + \\ & + \theta_{+(\bullet} \cup \theta_+ \cap \lambda_{L+} \cup \lambda_{R+\bullet)} + \lambda_{R+(\bullet} \cup \theta_+ \cap \lambda_{L+} \cup \theta_{+\bullet)} - \\ & - \theta_{+(\bullet} \cup \lambda_{R+} \cap \lambda_{L+} \cup \theta_{+\bullet)} + \theta_{+(\bullet} \cup \lambda_{R+} \cap \theta_+ \cup \lambda_{L+\bullet)}\end{aligned} \tag{52}$$
$$\Phi^{[\lambda^2\theta^2]}_{\texttt{mixed}} = \left(\lambda^{[\bullet}_{R+[\bullet} \, \theta^{\bullet]}_{+} \cap \lambda_{L+} \cup \theta_{+\bullet]}\right)^{\omega-\texttt{less}}_{\omega-\texttt{less}} + \left(\lambda_{L+} \frac{\partial}{\partial \theta_+}\right) \texttt{smth} \tag{53}$$

(Notice that the first term in $\Phi^{[\lambda^2\theta^2]}_{\texttt{mixed}}$ was called $\overset{0}{\Psi}$ in [4].)

$$\Phi^{[\lambda^3\theta]}_{\mathrm{up/dn}} = \Phi^{[\lambda^2\theta]}_{\mathrm{up/dn}} (\lambda_{L+} - \lambda_{R+})^{\bullet}_{\bullet} \tag{54}$$
$$\texttt{mod some equivalence relations}$$

$$\Phi^{[\lambda^3\theta^2]}_{\mathrm{up/dn/mixed}} = \Phi^{[\lambda^3\theta^2]}_{\mathrm{up/dn/mixed}} (\lambda_{L+} - \lambda_{R+})^{\bullet}_{\bullet} \tag{55}$$
$$\texttt{mod some equivalence relations}$$

$$\begin{aligned}
\Phi^{[\lambda^3\theta^3]}_{\rm up} &= \lambda^{(\bullet}_{R+} \cap \theta_+ \overset{0}{\cup} \lambda_{L+} \cap \theta_+ \cup \lambda_{L+} \cap \theta^{\bullet)}_+ + \\
&+ \theta^{(\bullet}_+ \cap \lambda_{L+} \cup \theta_+ \cap \lambda_{L+} \overset{0}{\cup} \theta_+ \cap \lambda^{\bullet)}_{R+} + \\
&+ \texttt{some } [\lambda^2_{R+}\lambda_{L+}\theta^3_+] \qquad (56)\\
\Phi^{[\lambda^3\theta^3]}_{\rm dn} &= \lambda_{R+(\bullet} \cup \theta_+ \overset{0}{\cap} \lambda_{L+} \cup \theta_+ \cap \lambda_{L+} \cup \theta_{+\bullet)} + \\
&+ \theta_{+(\bullet} \cup \lambda_{L+} \cap \theta_+ \cup \lambda_{L+} \overset{0}{\cap} \theta_+ \cup \lambda_{R+\bullet)} + \\
&+ \texttt{some } [\lambda^2_{R+}\lambda_{L+}\theta^3_+] \qquad (57)\\
\Phi^{[\lambda^3\theta^3]}_{\rm mixed1} &= \left((\lambda_{R+})^{(\bullet}_{[\bullet}\ \theta^{\bullet)}_+ \overset{0}{\cap} \lambda_{L+} \cup \theta_+ \cap \lambda_{L+} \cup \theta_{+\bullet]} \right)_{\omega-\text{less}} + \qquad (58)\\
&\quad + \texttt{ some } [\lambda^2_{R+}\lambda_{L+}\theta^3_+] \\
\Phi^{[\lambda^3\theta^3]}_{\rm mixed2} &= \left((\lambda_{R+})^{[\bullet}_{(\bullet}\ \theta^{\bullet]}_+ \overset{0}{\cap} \lambda_{L+} \cup \theta_+ \cap \lambda_{L+} \cup \theta_{+\bullet)} \right)^{\omega-\text{less}} + \qquad (59)\\
&\quad + \texttt{ some } [\lambda^2_{R+}\lambda_{L+}\theta^3_+]
\end{aligned}$$

$$\Phi^{[\lambda^3\theta^4]} = \frac{\partial}{\partial\theta}\ \Phi^{[\lambda^3\theta^5]} \qquad (60)$$

$$\begin{aligned}
&\Phi^{[\lambda^3\theta^5]} = \\
= &\left(1 - \frac{5}{3}\left(\lambda_R \frac{\partial}{\partial\lambda_L}\right)\right) ||[\theta_+ \cap \lambda_{L+} \cup \theta_+]_\Gamma \cap [\theta_+ \cup \lambda_{L+} \cap \theta_+]_\Gamma \cup \{\theta \overset{0}{\cap} \lambda_{L+}\}|| - \\
&- (\lambda_L \leftrightarrow \lambda_R) \qquad (61)
\end{aligned}$$

(See the symbolic computation in `L3T5Ansatz`, Section 4.4.2.)

Comments:

1. Notice that the free indices in $\Phi^{[\lambda^3\theta^3]}_{\rm up}$ and $\Phi^{[\lambda^3\theta^3]}_{\rm dn}$ are symmetrized; similar expressions with antisymmetrized indices is $Q^{(0)}$-exact

2. We found by a symbolic computation (`L3T5Derivatives` of Section 4.4.2) that:

$$\left(\Phi^{[\lambda^3\theta^3]}_{\rm up}\right)^{\alpha\beta} = -\frac{1}{288}\ \omega^{\alpha\alpha'}\omega^{\beta\beta'}\omega_{ab}\frac{\partial}{\partial\theta^{\alpha'}_a}\frac{\partial}{\partial\theta^{\beta'}_b}\Phi^{[\lambda^3\theta^5]} \qquad (62)$$

This implies that Eq. (34) defines a nonzero cohomology class.

3. The E-symmetry defines in Eq. (21) acts as follows:

$$E\Phi^{[\lambda^3\theta^5]} = -i\Phi^{[\lambda^3\theta^5]} \tag{63}$$

This agrees with:

$$Q\left(\Phi^{[\lambda^2\theta^2]}\,\Omega + \ldots\right) = \Phi^{[\lambda^3\theta^5]}\; t_m^2 t_m^2\Omega + \ldots \tag{64}$$

and $E\Phi^{[\lambda^2\theta^2]} = \Phi^{[\lambda^2\theta^2]}$ — see (23).

3.3 Resolution of the cohomology modules

Notice that the cohomology classes form a module over the commutative algebra $\mathcal{R} = \mathbb{C}[\lambda_L^1,\ldots,\lambda_L^{16},\;\lambda_R^1,\ldots,\lambda_R^{16}]/\Big((\lambda_L\Gamma^m\lambda_L) = (\lambda_R\Gamma^m\lambda_R) = 0\Big)$. Indeed, we can multiply any cohomology class by an arbitrary function of λ_L and λ_R, and get a new cohomology class. But this is not a free module. For example, one can see (the computation of `L3T2Eqs` of Section (4.4.2)) that $\lambda_L^\alpha\Phi^{[\lambda^2\theta^2]}_{\text{scalar}} = 0$ — this is a *relation*. It could be useful to classify such relations, and also the relations between relations, *etc.*

Mathematically, this can be formulated as follows. Let $H^{N,n}$ denote the cohomology of the ghost number N with the ghost number $N-n$ (*i.e.* with n thetas), and let $\ldots \to M_1^{(n)} \to M_0^{(n)} \to M^{(n)} \to 0$ denote the minimal free resolution of the $\mathcal{R}$-module $M^{(n)} = \bigoplus_{N,n} H^{N,n}$. For every i, the free module $M_i^{(n)}$ is a tensor product of $\mathcal{R}$ with some linear space μ_i on which $\mathbf{R}$ does not act:

$$M_i^{(n)} = \mu_i^{(n)} \otimes_{\mathbf{C}} \mathcal{R} \tag{65}$$

Notice that $\mu_i^{(n)}$ is a representation of $Spin(10)\times\mathbf{C}^\times$ — the group of rotations and rescaling of λ and θ_+. The dimensions of the $so(10)$-modules $\mu_i^{(n)}$ correspond to the numbers of generators of M_i. Having reformulated the problem in this mathematical language, we observe that it can be solved on a computer using `Macaulay2` [11].

The information about the free resolution is typically used to find the structure of the $Spin(10)\times\mathbf{C}^\times$-module on $\mu_i^{(n)}$ and therefore on $M^{(n)}$. We can proceed in the opposite direction, using the information about the structure of the $Spin(10)\times\mathbf{C}^\times$ -module on M to find the structure of μ_i using the formula:

$$\sum_i (-1)^i \mu_i \otimes \mathcal{S}^2 = \sum_N H^N \tau^N,$$

where $\mathcal{S}$ is a formal linear combination of $so(10)$ weights:

$$\mathcal{S} = \sum_{m=0}^{\infty} a_m \tau^m \tag{66}$$

$$a_m = [0, 0, 0, 0, m] \tag{67}$$

Let us define $\mathcal{S}^{-2} = \sum_{n=0}^{\infty} b_n \tau^n$, so that it satisfies:

$$(\mathcal{S}^2)(\mathcal{S}^{-2}) = (\sum_{m=0}^{\infty} a_m \tau^m)^{\otimes 2} \otimes (b_0 + b_1\tau + \cdots + b_i\tau^i + \cdots) = 1$$

We have $b_0 = 1$, $b_1 = -2a_1$, *etc.* Then we get:

$$\sum_i (-1)^i \mu_i = \sum_N H^N \tau^N \otimes \mathcal{S}^{-2}, \tag{68}$$

The analysis of the resolution of the cohomology module is given in the Sec. 4.3.2.

4 Details of calculation

4.1 Some vanishing theorems which follow immediately

The following classes are necessarily zero:

$$\lambda\theta^{\geq 3}\ ,\quad \lambda^2\theta^{\geq 5}\ ,\quad \lambda^3\theta^{\geq 6} \tag{69}$$

as follows from considering the term with the maximal power of λ_L.

4.2 Spectral sequence of a bicomplex

One method to compute the cohomology of $Q^{(0)}$ uses the spectral sequence of some bicomplex, which we will now describe.

4.2.1 Bicomplex

Introducing σ and d We will consider the spectral sequence corresponding to the following two differentials:

$$Q = Q_L + Q_R \tag{70}$$

$$\texttt{where } Q_L = \lambda_L^\alpha \frac{\partial}{\partial \theta_L^\alpha} \quad \texttt{and} \quad Q_R = \lambda_R^\alpha \frac{\partial}{\partial \theta_R^\alpha} \tag{71}$$

$$d = \sigma^\alpha \left(\frac{\partial}{\partial \theta_L^\alpha} - \frac{\partial}{\partial \theta_R^\alpha} \right) \tag{72}$$

Here σ^α is a new bosonic variable.

Z - grading All these differentials respect the total degree N:

$$N = \theta_L \frac{\partial}{\partial \theta_L} + \theta_R \frac{\partial}{\partial \theta_R} + \lambda_L \frac{\partial}{\partial \lambda_L} + \lambda_R \frac{\partial}{\partial \lambda_R} + \sigma \frac{\partial}{\partial \sigma} \tag{73}$$

Symmetry $\mathbf{Z}_2^{LR}$ Notice that both Q and d are invariant under the $\mathbf{Z}_2$ symmetry which exchanges $L \leftrightarrow R$ and $\sigma \to -\sigma$. We will call it $\mathbf{Z}_2^{LR}$.

Total complex We introduce the total complex, with the differential:

$$Q_{\text{tot}} = Q + d \tag{74}$$

It turns out that our problem is equivalent to calculating the cohomology of Q_{tot}. To prove this, we have to remember how the cohomology of the bicomplex is calculated using the spectral sequences.

4.2.2 Spectral sequence

First d then Q The first method is to first calculate the cohomology of d, and then consider Q as a small perturbation. The cohomology of d is:

$$H(d) = \text{Fun}(\ \theta_L + \theta_R\ ,\ \lambda_L\ ,\ \lambda_R\ ) \tag{75}$$

This gives the "first page" of the spectral sequence $\widetilde{E}$, *i.e.* $\widetilde{E}_1$. Notice that everything is graded by N. We get:

$$\widetilde{E}_1^{p>0,q}[N] = 0 \tag{76}$$

$$\widetilde{E}_1^{0,q}[N] = [\lambda_{L,R}^q (\theta_L + \theta_R)^{N-q}] \tag{77}$$

This implies that this spectral sequence terminates on the first page. Therefore:

- the cohomology of $Q+d$ is equal to the cohomology of Q on expressions which depend on $\theta_{L,R}$ only in the combination $\theta_L + \theta_R$

This is exactly what we want to calculate.

First Q **then** d The idea is to calculate first the cohomology of Q, and then act by d on it. We will develop this idea in Section 5.

4.3 Symbolic computations using `Macaulay2` and `LiE`

Another method is to use the symbolic computations. There are several tools which we will describe in this and the following section.

4.3.1 Computation of cohomology

Notations and setup Here we will use the description of the representations of $\mathfrak{so}(10)$ using weight diagramms:

$$[1,0,0,0,0] \quad \texttt{vector} \tag{78}$$

$$[0,1,0,0,0] \quad \texttt{antisymmetric 2-form} \tag{79}$$

$$[0,0,1,0,0] \quad \texttt{antisymmetric 3-form} \tag{80}$$

$$[0,0,0,1,0] \quad \texttt{antichiral spinor} \tag{81}$$

$$[0,0,0,0,1] \quad \texttt{chiral spinor} \tag{82}$$

We want to calculate the cohomology of the differential $Q = (\lambda_L + \lambda_R)\frac{\partial}{\partial\theta}$ where θ is an odd ten-dimensional spinor transforming according the representation $[0,0,0,0,1]$ of $\mathfrak{so}(10)$ and λ_L, λ_R are pure spinors transforming according the same representation. [4] More details about the computation procedure in this section could be referred to [12].

We will describe these cohomology groups as representations of the Lie algebra $\mathfrak{so}(10)$. Our calculations in this section use the computer programs

[4] As usual the representations are labeled by coordinates of their highest weight. The vector representation V has the highest weight $[1,0,0,0,0]$, the irreducible spinor representations have highest weights $[0,0,0,0,1]$, $[0,0,0,1,0]$.

`Macaulay2`[11] and `LiE` [13]. More precisely, we consider the differential Q acting on chain complex with components

$$\sum_{m_L=0}^{\infty}[0,0,0,0,m_L]\otimes\sum_{m_R=0}^{\infty}[0,0,0,0,m_R]\otimes\Lambda^n[0,0,0,0,1].$$

Here $[0,0,0,0,m_L]$ can be identified with the space of polynomial functions of the pure spinor λ_L of the order m_L, and $[0,0,0,0,m_R]$ with the space of polynomial functions of the pure spinor λ_R of the order m_R.

We apply the `LiE` program to obtain the decomposition of this complex into irreducible representations and use the dimensions of cohomology found by means of [11] to describe the action of the differential. The package `DGAlgebras` of `Macaulay2` already has procedures for calculating the cohomologies of the Koszul complex.

Results of computations The cohomology group has two gradings: $N = m_L + m_R + n$ and n:

$$H=\bigoplus_{N,n}H^{N,n} \tag{83}$$

Using LiE, we could explicitly describe the graded components of the cohomology group, $H^{N,n}$, with gradings by the following general formulas valid for $N\neq 4$:

$$\begin{aligned}
H^{N,0} &= [0,0,0,0,N] && (84)\\
H^{N,1} &= [1,0,0,0,N-2] && (85)\\
H^{N,2} &= [0,1,0,0,N-4] && (86)\\
H^{N,3} &= [0,0,1,0,N-6] && (87)\\
H^{N,4} &= [0,0,0,1,N-7] && (88)\\
H^{N,5} &= [0,0,0,0,N-8] && (89)
\end{aligned}$$

When $N = 4$, there is one additional term, a scalar, in $H^{4,2}$:

$$H^{4,2}=[0,0,0,0,0]\oplus[0,1,0,0,0] \tag{90}$$

The SO(10)-invariant part is in $H^{0,0}$, $H^{8,5}$, and $H^{4,2}$.

The dimensions of these cohomology groups are encoded in series $P_n(\tau) = \sum_N \dim H^{N,n}\tau^N$ (Poincaré series) that can be calculated by means of `Macaulay2` [11]:

$$
\begin{aligned}
P_0(\tau) &= \frac{1+5\tau+5\tau^2+\tau^3}{(1-\tau)^{11}} \\
P_1(\tau) &= \frac{10\tau^2+34\tau^3+16\tau^4)}{(1-\tau)^{11}} \\
P_2(\tau) &= (46\tau^4+54\tau^5+66\tau^6-166\tau^7+330\tau^8-462\tau^9+462\tau^{10}-330\tau^{11} \\
&\quad +165\tau^{12}-55\tau^{13}+11\tau^{14}-\tau^{15})/(1-\tau)^{11} \\
P_3(\tau) &= (120\tau^6-120\tau^7+330\tau^8-462\tau^9+462\tau^{10}-330\tau^{11}+165\tau^{12} \\
&\quad -55\tau^{13}+11\tau^{14}-\tau^{15})/(1-\tau)^{11} \\
P_4(\tau) &= \frac{16\tau^7+34\tau^8+10\tau^9}{(1-\tau)^{11}} \\
P_5(\tau) &= \frac{\tau^8+5\tau^9+5\tau^{10}+\tau^{11}}{(1-\tau)^{11}}
\end{aligned}
$$

The cohomology $H^n = \bigoplus_N H^{N,n}$ can be regarded as a $\mathbb{C}[\lambda_L^1,\dots,\lambda_L^{16},\ \lambda_R^1,\dots,\lambda_R^{16}]$-module. Using `Macaulay2` one can obtain the number of its generators. The number of $0,\cdots,5$-th cohomology generators are $1,10,46,120,16,1$, respectively. Using the highest weight vector representation, beside of 1, they are

$$H^{2,1} = [1,0,0,0,0], \tag{91}$$
$$H^{4,2} = [0,0,0,0,0]+[0,1,0,0,0], \tag{92}$$
$$H^{6,3} = [0,0,1,0,0], \tag{93}$$
$$H^{7,4} = [0,0,0,1,0], \tag{94}$$
$$H^{8,5} = [0,0,0,0,0]. \tag{95}$$

The expressions for generators are given in Sec.3.1 , specifically:

$H^{2,1}$	$[1,0,0,0,0]$	$\Phi^{[\lambda\theta]}$
$H^{4,2}$	$[0,0,0,0,0]$	$\Phi^{[\lambda^2\theta^2]}_{\text{scalar}}$
	$[0,1,0,0,0]$	$\Phi^{[\lambda^2\theta^2]}_{2-\text{form}}$
$H^{6,3}$	$[0,0,1,0,0]$	$\Phi^{[\lambda^3\theta^3]}$
$H^{7,4}$	$[0,0,0,1,0]$	$\Phi^{[\lambda^3\theta^4]}$
$H^{8,5}$	$[0,0,0,0,0]$	$\Phi^{[\lambda^3\theta^5]}$

Generating cohomology by differentiation with respect to θ Some generators can be obtained from the generator of $H^{8,5}$ (denoted later by Ψ) by means of differentiation with respect to θ. Namely, the generators belonging to $H^{7,4}$ are equal to $\frac{\partial\Psi}{\partial\theta^\alpha}$, the generators belonging to $H^{6,3}$ are equal to $\Gamma'^{\alpha\beta}_{abc}\frac{\partial^2}{\partial\theta^\alpha\partial\theta^\beta}\Psi$, where Γ' is some matrix anti-symmetric with respect to a, b, c, and not symmetric with respect to α, β. One can find the minimal number of generators having the property that all other generators can be obtained from them by means of differentiation with respect to θ. To calculate this number we notice, that the cohomology can be considered also as module over the ring $\mathbb{C}[\lambda_L^1, \dots, \lambda_L^{16},\ \lambda_R^1, \dots, \lambda_R^{16}] \otimes \Lambda[b]$ where b_α stands for $\frac{\partial}{\partial\theta^\alpha}$. Calculations with `Macaulay2` allow us to calculate the number of generators of this module.

It is equal to 58. This means that the module we are interested in is generated by $1, H^{2,1} = [1,0,0,0,0], H^{4,2} = [0,0,0,0,0] + [0,1,0,0,0]$, and $H^{8,5} = [0,0,0,0,0]$.

Behaviour under the exchange $\lambda_L \leftrightarrow \lambda_R$ Notice that our differential is invariant with respect to the involution $\lambda_L \to \lambda_R$. Therefore this involution acts on homology. The cohomology groups $H^{k,0}$ and $H^{k,2}$ (except $[0,0,0,0,0]$ in $H^{4,2}$) are invariant (even) with respect to this involution. Other cohomology groups are odd (*i.e.* the involution acts as multiplication by -1). In particular, the generators 1 and $[0,1,0,0,0]$ in $H^{4,2}$ are even, other generators are odd.

4.3.2 Resolution of the cohomology modules

Based on the method discussed in Sec. 3.3, one can find a minimal free resolution of the $\mathcal{R}$-module $M = \sum_N H^{N,n}$, where $\mathcal{R} = \mathbb{C}[\lambda_L^1, \dots, \lambda_L^{16},\ \lambda_R^1, \dots, \lambda_R^{16}] = \sum_{m_L=0}^{\infty} [0,0,0,0,m_L] \otimes \sum_{m_R=0}^{\infty} [0,0,0,0,m_R]$. The reader may wish to consult [14] on this subject. The free resolution has the form

$$\cdots \to M_i \to \cdots \to M_0 \to M \to 0$$

where $M_i = \mu_i \otimes \mathcal{R}$, and
μ_0 - generators of M;
μ_1 - relations between generators of M;
μ_2 - relations between relations ;
$\dots$

We give the structure of μ_i as $so(10)$-module.

- $n = 0$,

$$\mu_0 = [0,0,0,0,0], \dim(\mu_0) = 1, \deg(\mu_0) = 0;$$

$$\mu_1 = [0,0,0,0,1], \dim(\mu_1) = 16, \deg(\mu_1) = 1;$$

$$\mu_2 = [0,0,1,0,0] + [1,0,0,0,0], \dim(\mu_2) = 130, \deg(\mu_2) = 2;$$

$$\mu_3 = 2\times[0,0,0,1,0]+[0,1,0,1,0]+[1,0,0,0,1], \dim(\mu_3) = 736, \deg(\mu_3) = 3;$$

$$\begin{aligned}\mu_4 =& 2\times[0,0,0,0,0] + 2\times[0,0,0,1,1] + 3\times[0,1,0,0,0] + [0,2,0,0,0]+\\ &+ [1,0,0,2,0] + [1,0,1,0,0] + [2,0,0,0,0], \dim(\mu_4) = 3376, \deg(\mu_4) = 4;\end{aligned}$$

$$\cdots$$

- $n = 1$,

$$\mu_0 = [1,0,0,0,0], \dim(\mu_0) = 10, \deg(\mu_0) = 2;$$

$$\mu_1 = 2\times[0,0,0,1,0] + [1,0,0,0,1], \dim(\mu_1) = 176, \deg(\mu_1) = 3;$$

$$\begin{aligned}\mu_2 =& 3\times[0,0,0,0,0] + 2\times[0,0,0,1,1] + 4\times[0,1,0,0,0] + [1,0,1,0,0]+\\ &+ [2,0,0,0,0], \dim(\mu_2) = 1602, \deg(\mu_2) = 4;\end{aligned}$$

$$\begin{aligned}\mu_3 =& 8\times[0,0,0,0,1] + 2\times[0,0,1,1,0] + 4\times[0,1,0,0,1] + 8\times[1,0,0,1,0]+\\ &+ [1,1,0,1,0] + [2,0,0,0,1], \dim(\mu_3) = 10336, \deg(\mu_3) = 5;\end{aligned}$$

$$\cdots$$

- $n = 2$,

$$\mu_0 = [0,0,0,0,0] + [0,1,0,0,0], \dim(\mu_0) = 46, \deg(\mu_0) = 4;$$

$$\mu_1 = 4\times[0,0,0,0,1]+[0,1,0,0,1]+2\times[1,0,0,1,0], \dim(\mu_1) = 912, \deg(\mu_1) = 5;$$

$$\begin{aligned}\mu_2 =& 3\times[0,0,0,0,2] + 3\times[0,0,0,2,0] + 8\times[0,0,1,0,0]+\\ &+ [0,1,1,0,0] + 9\times[1,0,0,0,0] + 2\times[1,0,0,1,1]+\\ &+ 4\times[1,1,0,0,0], \dim(\mu_2) = 9512, \deg(\mu_2) = 6;\end{aligned}$$

$$\cdots$$

- $n = 3$,

$$\mu_0 = [0,0,1,0,0], \dim(\mu_0) = 120, \deg(\mu_0) = 6;$$

$$\begin{aligned}\mu_1 =& 2\times[0,0,0,1,0] + [0,0,1,0,1] + 2\times[0,1,0,1,0] + 2\times[1,0,0,0,1],\\ &\dim(\mu_1) = 2640, \deg(\mu_1) = 7;\end{aligned}$$

$$\begin{aligned}\mu_2 =& 3\times[0,0,0,0,0] + 8\times[0,0,0,1,1] + [0,0,2,0,0] + 7\times[0,1,0,0,0]+\\ &+ 2\times[0,1,0,1,1] + 3\times[0,2,0,0,0] + 2\times[1,0,0,0,2] + 3\times[1,0,0,2,0]+\\ &+ 5\times[1,0,1,0,0] + 3\times[2,0,0,0,0], \dim(\mu_2) = 30450, \deg(\mu_2) = 8;\end{aligned}$$

$$\cdots$$

- $n = 4$,

$$\mu_0 = [0,0,0,1,0], \dim(\mu_0) = 16, \deg(\mu_0) = 7;$$

$$\mu_1 = 2\times[0,0,0,0,0]+[0,0,0,1,1]+2\times[0,1,0,0,0], \dim(\mu_1) = 302, \deg(\mu_1) = 8;$$

$$\begin{aligned}\mu_2 =& 5\times[0,0,0,0,1] + [0,0,1,1,0] + 2\times[0,1,0,0,1] + 4\times[1,0,0,1,0],\\ &\dim(\mu_2) = 2976, \deg(\mu_2) = 9;\end{aligned}$$

$$\begin{aligned}\mu_3 =& 3\times[0,0,0,0,2] + 6\times[0,0,0,2,0] + 10\times[0,0,1,0,0] + [0,1,0,2,0]+\\ &+ 2\times[0,1,1,0,0] + 10\times[1,0,0,0,0] + 4\times[1,0,0,1,1]+\\ &+ 6\times[1,1,0,0,0], \dim(\mu_3) = 20902, \deg(\mu_3) = 10;\end{aligned}$$

$$\begin{aligned}\mu_4 =& 24\times[0,0,0,1,0] + 6\times[0,0,0,2,1] + 8\times[0,0,1,0,1]+\\ &+ 23\times[0,1,0,1,0] + 2\times[0,2,0,1,0] + 24\times[1,0,0,0,1]+\\ &+ [1,0,0,3,0] + 4\times[1,0,1,1,0] + 6\times[1,1,0,0,1]+\\ &+ 9\times[2,0,0,1,0], \dim(\mu_4) = 120224, \deg(\mu_4) = 11;\end{aligned}$$

$$\cdots$$

- $n = 5$,

$$\mu_0 = [0,0,0,0,0], \dim(\mu_0) = 1, \deg(\mu_0) = 8;$$

$$\mu_1 = [0,0,0,0,1], \dim(\mu_1) = 16, \deg(\mu_1) = 9;$$

$$\mu_2 = [0,0,1,0,0] + [1,0,0,0,0], \dim(\mu_2) = 130, \deg(\mu_2) = 10;$$

$$\mu_3 = 2\times[0,0,0,1,0]+[0,1,0,1,0]+[1,0,0,0,1], \dim(\mu_3) = 736, \deg(\mu_3) = 11;$$

$$\begin{aligned}\mu_4 =& 2\times[0,0,0,0,0] + 2\times[0,0,0,1,1] + 3\times[0,1,0,0,0] + [0,2,0,0,0]+\\ &+ [1,0,0,2,0] + [1,0,1,0,0] + [2,0,0,0,0], \dim(\mu_4) = 3376, \deg(\mu_4) = 12;\end{aligned}$$

$$\cdots$$

where $i\times[a,b,c,d,e]$ denotes the representation $[a,b,c,d,e]$ with multiplicity i.

4.4 Symbolic computations using a canonical form of a pair of pure spinors

4.4.1 Geometry of a pair of pure spinors in $AdS_5 \times S^5$

As was explained in [4], the orbit of the generic pure spinor under $so(5) \oplus so(5) = sp(2) \oplus sp(2)$ is a cone over the group manifold of the symplectic group $Sp(2)$:

$$(\lambda_{R+})^\alpha_a \omega^{ab} (\lambda_{R+})^\beta_b \;= \frac{1}{4} ||\lambda_{R+} \cap \lambda_{R+}|| \; \omega^{\alpha\beta} \tag{96}$$

$$(\lambda_{R+})^\alpha_a \omega_{\alpha\beta} (\lambda_{R+})^\beta_b \;= \frac{1}{4} ||\lambda_{R+} \cap \lambda_{R+}|| \; \omega_{ab} \tag{97}$$

(These two equations are equivalent when $||\lambda_{R+} \cap \lambda_{R+}|| \neq 0$.) The same constraints are imposed on λ_{L+}.

Let us consider the orbit with $||\lambda_{R+} \cap \lambda_{R+}|| = 4c_R$ where c_R is a complex number. In this case we can use the change of variables:

$$(\lambda_{R+})^\alpha_a \mapsto M^b_a (\lambda_{R+})^\alpha_b \tag{98}$$

where $M \in Sp(2)$, to "diagonalize" λ_{R+}:

$$(\lambda_{R+})^\alpha_a = c_R \delta^\alpha_a \tag{99}$$

This choice of λ_{R+} breaks $sp(2) \oplus sp(2)$ to a diagonal $sp(2)$.

In a generic case, this residual $sp(2)$ can be used to bring λ_{L+} to the form:

$$\lambda_{L+} = c_L \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a^{-1} & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b^{-1} \end{pmatrix} \tag{100}$$

This is only true when λ_{L+} is in generic position with respect to λ_{R+}. This is because a *generic* quadratic Hamiltonian $H(q_1, p_1, q_2, p_2)$ can be brought to the form $\alpha q_1 p_1 + \beta q_2 p_2$ by a canonical transformation[5].

We used this explicit parametrization of the pair of pure spinors in our computer calculation: see Section 4.4.2.

We conclude that there is a 4-parameter family of the orbits of maximal dimension $22 - 4 = 18$. Notice that the dimension of $sp(2) \oplus sp(2)$ is 20,

[5] in this particular calculation we do not care about reality

therefore there should be $20-18=2$ elements of the diagonal part of $sp(2)\oplus sp(2)$ stabilizing the pair $\lambda_{L+},\lambda_{R+}$. They are parametrized by ν_1 and ν_2:

$$\begin{pmatrix} \nu_1 & 0 & 0 & 0 \\ 0 & -\nu_1 & 0 & 0 \\ 0 & 0 & \nu_2 & 0 \\ 0 & 0 & 0 & -\nu_2 \end{pmatrix} \in \; sp(2) \tag{101}$$

Notice under that the action of $Spin(10)\times \mathbf{C}^{\times}\times \mathbf{C}^{\times}$, any two *generic* pairs $(\lambda_L^{(1)},\lambda_R^{(1)})$ and $(\lambda_L^{(2)},\lambda_R^{(2)})$ are equivalent, *i.e.* there are no invariants. But if we restrict to $so(5)\oplus so(5)$, then there are invariants c_L, c_R, a, b.

4.4.2 Computer program for direct calculation of cohomology

We have a computer program which does symbolic manipulations with elements of free supercommutative algebras. This allows a straightforward symbolic computation of the BRST cohomology. The program is available here:

`http://code.google.com/p/minitheta/w/list`

The idea is to straightforwardly compute the cohomology of the operator $(\lambda_{L+}+\lambda_{R+})\frac{\partial}{\partial\theta_+}$ on expressions polynomial in $\lambda_{L+},\lambda_{R+},\theta_+$. We write all the possible $sp(2)\oplus sp(2)$-covariant polynomials which can be constructed from $\lambda_{L+},\lambda_{R+},\theta_+$ and the symplectic forms $\omega_{\alpha\beta}$ and ω^{ab}, and then compute the action of $(\lambda_{L+}+\lambda_{R+})\frac{\partial}{\partial\theta_+}$ on the space of these polynomials. The pure spinor constraints on λ_{L+} and λ_{R+} are taken into account by subsitution of expressions (100) and (99) for λ_{L+} and λ_{R+}.

5 Use of spectral sequence

In this section we will compute the cohomology using the spectral sequence for the bicomplex $Q+d$ of Section 4.2. We have seen in Section 4.2 that the cohomology of $Q+d$ is the same as the cohomology of our $Q^{(0)}$. Here we will calculate this cohomology by first computing the cohomology of Q (which is well known) and then treating d as a perturbation.

The cohomology of Q is well known:

$$H(Q) = \mathrm{Fun}\Big(\sigma,\, [\lambda_L\theta_L],\; [\lambda_L\theta_L^2],\; [\lambda_L^2\theta_L^3],\; [\lambda_L^2\theta_L^4],\; [\lambda_L^3\theta_L^5],$$
$$[\lambda_R\theta_R],\; [\lambda_R\theta_R^2],\; [\lambda_R^2\theta_R^3],\; [\lambda_R^2\theta_R^4],\; [\lambda_R^3\theta_R^5]\Big) \;/\; \mathrm{Im}(Q) \quad (102)$$

We identify:

$$E_1^{p,q}[N] = \texttt{ elements of the form } [\sigma^p\lambda^q\theta^{N-p-q}] \texttt{ in } H(Q) \quad (103)$$

As we explained in Section 4.2, N commutes with both Q and d.

5.1 Case $N = 0$

We have $H(Q)[0] = \mathbf{C}$ — just constants. In this case d acts trivially and we get:

$$H(d+Q)[0] = \mathbf{C} \quad (104)$$

5.2 Case $N = 1$

We observe that $H(Q)[1]$ is generated by σ:

$$E_1^{1,0}[1] = \mathbf{C} \texttt{ generated by } \sigma \quad (105)$$
$$E_1^{0,1}[1] = 0 \quad (106)$$

and all other $E_1^{p,q} = 0$. Therefore $E_\infty^1 = \widetilde{E}_\infty^1 = \mathbf{C}$, in fact $\widetilde{E}_\infty^1$ is generated by $\lambda_L - \lambda_R$. Indeed, $\lambda_L - \lambda_R$ is Q-closed and cannot be obtained as Q of an expression involving θ_L and θ_R only through $\theta_L + \theta_R$. Therefore, it represents a nontrivial cohomology class:

$$H(d+Q)[1] = \mathbf{C} \;\texttt{ generated by either } \sigma \texttt{ or } \lambda_L - \lambda_R$$
$$\texttt{depending on the point of view} \quad (107)$$

5.3 Case $N = 2$

Notice that $H(Q)[2]$ is generated by the following elements:

$$E_1^{0,0}[2] = E_1^{1,0}[2] = 0 \tag{108}$$
$$E_1^{0,1}[2] = V \oplus V \texttt{ generated by } (\lambda_L \Gamma^m \theta_L) \texttt{ and } (\lambda_R \Gamma^m \theta_R) \tag{109}$$
$$E_1^{2,0}[2] = S^2 \mathcal{S} \texttt{ generated by } \sigma^\alpha \sigma^\beta \tag{110}$$
$$E_1^{1,1}[2] = 0 \tag{111}$$
$$E_1^{0,2}[2] = 0 \tag{112}$$

We observe that $d_1 : E_1^{p,q}[2] \to E_1^{p+1,q}[2]$ is zero, therefore $E_2^{p,q}[2] = E_1^{p,q}[2]$. However, the d_2 acts nontrivially:

$$d_2 \; : \; E_2^{0,1}[2] \to E_2^{2,0}[2] \tag{113}$$
$$d_2\left((\lambda_L \Gamma^m \theta_L) + (\lambda_R \Gamma^m \theta_R)\right) = (\sigma \Gamma^m \sigma) \tag{114}$$
$$d_2\left((\lambda_L \Gamma^m \theta_L) - (\lambda_R \Gamma^m \theta_R)\right) = 0 \tag{115}$$

We conclude that d_2 cancels $(\lambda_L \Gamma^m \theta_L) + (\lambda_R \Gamma^m \theta_R)$ against $(\sigma \Gamma^m \sigma)$ and we are left with the following:

- $E_\infty[2]$ is generated by $((\lambda_L - \lambda_R)\Gamma^m(\theta_L + \theta_R))$ and $(\sigma \Gamma^{m_1 \dots m_5} \sigma)$
- we also observe that $(\sigma \Gamma^{m_1 \dots m_5} \sigma)$ is $d + Q$-equivalent to:

$$((\lambda_L - \lambda_R)\Gamma^{m_1 \dots m_5}(\lambda_L - \lambda_R)) \tag{116}$$

The expression $((\lambda_L - \lambda_R)\Gamma^{m_1 \dots m_5}(\lambda_L - \lambda_R))$ is in cohomology, in a sense that it cannot be obtained as Q of something which only depends on $\theta_L + \theta_R$.

5.4 Case $N = 3$

5.4.1 $E_1^{p,q}[3]$

$H(Q)[3]$ is generated by:

$$E_1^{0,0}[3] = E_1^{1,0}[3] = 0 \tag{117}$$

$$E_1^{0,1}[3] = \mathcal{S}' \oplus \mathcal{S}' \texttt{ generated by } \lambda_L\theta_L^2 \text{ and } \lambda_R\theta_R^2 \tag{118}$$

$$E_1^{0,2}[3] = E_1^{2,0}[3] = 0 \tag{119}$$

$$\begin{aligned} E_1^{1,1}[3] = \mathcal{S} \otimes (V \oplus V) \texttt{ generated by} \\ \sigma^\alpha(\theta_L\Gamma^m\lambda_L) \texttt{ and } \sigma^\alpha(\theta_R\Gamma^m\lambda_R) \end{aligned} \tag{120}$$

$$E_1^{3,0}[3] = S^3\mathcal{S} \texttt{ generated by } \sigma^\alpha\sigma^\beta\sigma^\gamma \tag{121}$$

$$E_1^{2,1}[3] = E_1^{1,2}[3] = E_1^{0,3}[3] = 0 \tag{122}$$

The differential $d_1 : E_1^{p,q}[3] \to E_1^{p+1,q}[3]$ acts nontrivially in the following components:

$$\begin{aligned} d_1 &: E_1^{0,1}[3] \to E_1^{1,1}[3] \\ d_1\left((\lambda_L\Gamma_m\theta_L)\Gamma^m\theta_L\right) &\simeq (\lambda_L\Gamma_m\theta_L)\Gamma^m\sigma \qquad (123) \\ d_1\left((\lambda_R\Gamma_m\theta_R)\Gamma^m\theta_R\right) &\simeq (\lambda_R\Gamma_m\theta_R)\Gamma^m\sigma \qquad (124) \end{aligned}$$

5.4.2 $E_2^{p,q}[3]$

Eqs. (123) and (124) imply that:

$$E_2^{0,1}[3] = 0 \tag{125}$$

$$\begin{aligned} E_2^{1,1}[3] = (\mathcal{S} \otimes (V \oplus V))/(\mathcal{S}' \oplus \mathcal{S}') \texttt{ where } \mathcal{S}' \oplus \mathcal{S}' \\ \texttt{is generated by } (\lambda_L\Gamma_m\theta_L)\Gamma^m\sigma \texttt{ and } (\lambda_R\Gamma_m\theta_R)\Gamma^m\sigma \end{aligned} \tag{126}$$

and the other components of $E_2[3]$ are the same as the corresponding components of $E_1[3]$. In particular, $E_2^{3,0}[3] = E_1^{3,0}[3] = S^3\mathcal{S}$.

5.4.3 $E_3^{p,q}[3]$

There is a nontrivial d_2:

$$d_2 \;:\; E_2^{1,1}[3] \to E_2^{3,0}[3] \tag{127}$$

$$d_2\; (\sigma^\alpha(\theta_L\Gamma^m\lambda_L)) = \sigma^\alpha(\sigma\Gamma^m\sigma) \tag{128}$$

$$d_2\; (\sigma^\alpha(\theta_R\Gamma^m\lambda_R)) = \sigma^\alpha(\sigma\Gamma^m\sigma) \tag{129}$$

(The overall coefficient may be wrong, but the relative sign is as it should be.) This implies:

$$E_3^{1,1}[3] = (\mathcal{S}\otimes V)/\mathcal{S}' \texttt{ generated by } \sigma^\alpha\left((\theta_L\Gamma^m\lambda_L) - (\theta_R\Gamma^m\lambda_R)\right) \quad (130)$$
$$E_3^{3,0}[3] = S^3\mathcal{S}/(\mathcal{S}\otimes V) \quad (131)$$

In the language of $\tilde{E}$, these two components correspond to:

$$(\lambda_L^\alpha - \lambda_R^\alpha)\left((\theta_L+\theta_R)\Gamma^m(\lambda_L-\lambda_R)\right) \quad (132)$$
$$\texttt{and something cubic in } (\lambda_L - \lambda_R) \quad (133)$$

There are some equivalence relations. If we contract in (132) the indices m and α with the gamma-matrix, the resulting expression will be Q-exact:

$$\begin{aligned}&\Gamma_m(\lambda_L+\lambda_R)\left((\theta_L+\theta_R)\Gamma^m(\lambda_L+\lambda_R)\right)\\ &= Q\left(\Gamma_m(\theta_L+\theta_R)\left((\theta_L+\theta_R)\Gamma^m(\lambda_L+\lambda_R)\right)\right)\end{aligned} \quad (134)$$

There must be a similar equivalence relation in (133), which we did not study.

5.5 Case $N = 4$

5.5.1 $E_1^{p,q}[4]$ and d_1

$$E_1^{0,0}[4] = 0 \quad (135)$$
$$E_1^{1,0}[4] = E_1^{0,1}[4] = E_1^{2,0}[4] = 0 \quad (136)$$
$$\begin{aligned}E_1^{1,1}[4] &= (\mathcal{S}\otimes\mathcal{S}')\oplus(\mathcal{S}\otimes\mathcal{S}') \texttt{ generated by}\\ &\quad \sigma^\alpha(\theta_L^2\lambda_L)_\beta \texttt{ and } \sigma^\alpha(\theta_R^2\lambda_R)_\beta\end{aligned} \quad (137)$$
$$E_1^{0,2}[4] = V\otimes V \texttt{ generated by } (\theta_L\Gamma^m\lambda_L)(\theta_R\Gamma^n\lambda_R) \quad (138)$$
$$E_1^{3,0}[4] = 0 \quad (139)$$
$$\begin{aligned}E_1^{2,1}[4] &= (S^2\mathcal{S})\otimes(V\oplus V) \texttt{ generated by}\\ &\quad \sigma^\alpha\sigma^\beta(\theta_L\Gamma^m\lambda_L) \texttt{ and } \sigma^\alpha\sigma^\beta(\theta_R\Gamma^m\lambda_R)\end{aligned} \quad (140)$$
$$E_1^{1,2}[4] = E_1^{0,3}[4] = 0 \quad (141)$$
$$E_1^{4,0}[4] = S^4\mathcal{S} \texttt{ generated by } \sigma^\alpha\sigma^\beta\sigma^\gamma\sigma^\delta \quad (142)$$
$$E_1^{3,1}[4] = E_1^{2,2}[4] = 0 \quad (143)$$
$$E_1^{1,3}[4] = E_1^{0,4}[4] = 0 \quad (144)$$

There is a nontrivial d_1:

$$d_1 \ : \ E_1^{1,1}[4] \to E_1^{2,1}[4], \ \texttt{all other components zero} \tag{145}$$

$$d_1 \left(\sigma^\alpha(\theta_L^2\lambda_L)_\beta\right) = \sigma^\alpha \left(\sigma\Gamma^m(\theta_L\Gamma_m\lambda_L)\right)_\beta \tag{146}$$

$$d_1 \left(\sigma^\alpha(\theta_R^2\lambda_R)_\beta\right) = \sigma^\alpha \left(\sigma\Gamma^m(\theta_R\Gamma_m\lambda_R)\right)_\beta \tag{147}$$

Notice that $\ker\left(d_1 : E_1^{1,1}[4] \to E_1^{2,1}[4]\right) = 0$, therefore $E_2^{1,1}[4] = 0$.

5.5.2 $E_2^{p,q}$ and d_2

Eq. (145) implies that all the components of $E_2[4]$ are the same as the corresponding components of $E_1[4]$, except for:

$$E_2^{1,1}[4] = 0 \tag{148}$$

$$E_2^{2,1}[4] = \frac{(S^2\mathcal{S}) \otimes (V \oplus V)}{(\mathcal{S}\otimes\mathcal{S}') \oplus (\mathcal{S}\otimes\mathcal{S}'\ )} \tag{149}$$

The following components of $d_2 \ : \ E_2^{p,q} \to E_2^{p+2,q-1}$ are potentially nonzero:

$$d_2 \ : E_2^{0,2}[4] \to E_2^{2,1}[4] \tag{150}$$

$$d_2 \ : E_2^{2,1}[4] \to E_2^{4,0}[4] \tag{151}$$

5.5.3 Calculation of $d_2 \ : \ E_2^{0,2}[4] \to E_2^{2,1}[4]$

$$\begin{aligned} d\Big((\lambda_L\Gamma^m\theta_L)(\lambda_R\Gamma^n\theta_R)\Big) &= (\lambda_L\Gamma^m\sigma)(\lambda_R\Gamma^n\theta_R) + (\lambda_L\Gamma^m\theta_L)(\lambda_R\Gamma^n\sigma) = \\ &= Q\Big((\theta_L\Gamma^m\sigma)(\lambda_R\Gamma^n\theta_R) + (\theta_R\Gamma^n\sigma)(\lambda_L\Gamma^m\theta_L)\Big) \end{aligned} \tag{152}$$

$$\begin{aligned} & d\Big((\theta_L\Gamma^m\sigma)(\lambda_R\Gamma^n\theta_R) + (\theta_R\Gamma^n\sigma)(\lambda_L\Gamma^m\theta_L)\Big) = \\ & = (\sigma\Gamma^m\sigma)(\lambda_R\Gamma^n\theta_R) - (\sigma\Gamma^n\sigma)(\lambda_L\Gamma^m\theta_L) - Q\Big((\sigma\Gamma^m\theta_L)(\theta_R\Gamma^n\sigma)\Big) \end{aligned} \tag{153}$$

In order for the d_2 to vanish, this should be in the image of d_1, where d_1 is given by Eqs. (146) and (147). It is immediately clear that d_2 annihilates the scalar component of $E_2^{0,2}[4] = V \otimes V$:

$$d_2\Big(\mathbf{C} \subset V \otimes V\Big) = 0 \tag{154}$$

This means that the cohomology class of $(\theta_L\Gamma^m\lambda_L)(\theta_R\Gamma_m\lambda_R)$ can be represented by an expression depending only on $\theta_L+\theta_R$, as was demonstrated in Appendix A2 of `arXiv:1105.2231`.

Also d_2 annihilates the antisymmetric tensor:

$$d_2\Big((\lambda_L\Gamma^{[m}\theta_L)(\lambda_R\Gamma^{n]}\theta_R)\Big)=0 \tag{155}$$

which corresponds to (30). Indeed:

$$d_1\Big((\theta_L^2\lambda_L)_\beta(\Gamma^{mn})_\alpha^\beta\sigma^\alpha\Big)=(\theta_L\Gamma_p\lambda_L)(\sigma\Gamma^p\Gamma^{mn}\sigma)\simeq(\theta_L\Gamma_{[m}\lambda_L)(\sigma\Gamma_{n]}\sigma) \tag{156}$$

and this covers (153).

5.5.4 Calculation of $d_2\ :\ E_2^{2,1}[4]\to E_2^{4,0}[4]$

This is essentially the same calculation as $d_2\ :\ E_2^{0,1}[2]\to E_2^{2,0}[2]$. We have $E_2^{4,0}=S^4\mathcal{S}$ and

$$d_2\left(E_2^{2,1}[4]\right)=S^2\mathcal{S}\otimes V\subset S^4\mathcal{S} \tag{157}$$

This implies:

$$E_3^{0,2}[4]=\mathbf{C}\ \texttt{(this is }(\theta_L\Gamma^m\lambda_L)(\theta_R\Gamma_m\lambda_R)\texttt{)} \tag{158}$$

$$E_3^{2,1}[4]=\frac{(S^2\mathcal{S})\otimes V}{(\mathcal{S}\otimes(\mathcal{S}'\oplus\mathcal{S}'))\ \oplus\ (V\otimes V)_0} \tag{159}$$

$$E_3^{4,0}[4]=\frac{S^4\mathcal{S}}{S^2\mathcal{S}\otimes V} \tag{160}$$

The d_3 is zero, therefore $H(d+Q)[4]=E_3[4]$.

Notice that $E_3^{2,1}[4]$ corresponds to something like $(\lambda_L-\lambda_R)^2((\theta_L+\theta_R)\Gamma^m(\lambda_L-\lambda_R))$, and $E_3^{4,0}[4]$ to something quartic in $\lambda_L-\lambda_R$. In both cases, there are some equivalence relations which we did not calculate.

5.5.5 Symbolic computations

See Section 4.4.2. Computation in `L2T2Eqs` show the nontrivial cohomology class:

$$\begin{gathered}\lambda_{L+}^{(\bullet}\cap\theta_+\cup\theta_+\cap\lambda_{R+}^{\bullet)}+\lambda_{L+}^{(\bullet}\cap\lambda_{R+}\cup\theta_+\cap\theta_+^{\bullet)}+\theta_+^{(\bullet}\cap\theta_+\cup\lambda_{L+}\cap\lambda_{R+}^{\bullet)}+\\+\lambda_{R+}^{(\bullet}\cap\theta_+\cup\lambda_{L+}\cap\theta_+^{\bullet)}-\theta_+^{(\bullet}\cap\lambda_{R+}\cup\lambda_{L+}\cap\theta_+^{\bullet)}+\theta_+^{(\bullet}\cap\lambda_{R+}\cup\theta_+\cap\lambda_{L+}^{\bullet)}\end{gathered} \tag{161}$$

This is the part of (30) with both indices inside AdS. Also notice:

$$\left(\{\lambda_L \overset{0}{\cap} \theta\} \cup \{\lambda_R \overset{0}{\cap} \theta\} - \{\lambda_R \overset{0}{\cap} \theta\} \cup \{\lambda_L \overset{0}{\cap} \theta\}\right)_{\omega-\texttt{less}} = 0 \tag{162}$$

5.6 Case $N = 5$

5.6.1 $E_1^{p,q}[5]$

$$E_1^{0,0}[5] = 0 \tag{163}$$

$$E_1^{1,0}[5] = E_1^{0,1}[5] = 0 \tag{164}$$

$$E_1^{2,0}[5] = E_1^{1,1}[5] = 0 \tag{165}$$

$$\begin{aligned} E_1^{0,2}[5] = {} & (\mathcal{S}' \otimes V) \oplus (V \otimes \mathcal{S}') \texttt{ generated by} \\ & (\theta_L^2\lambda_L)_\alpha(\theta_R\lambda_R)^m\ ,\ (\theta_L\lambda_L)^m(\theta_R^2\lambda_R)_\alpha \texttt{ and} \\ & (\theta_L^3\lambda_L^2)_\alpha\ ,\ (\theta_R^3\lambda_R^2)_\alpha \end{aligned} \tag{166}$$

$$E_1^{3,0}[5] = E_1^{0,3}[5] = 0 \tag{167}$$

$$\begin{aligned} E_1^{2,1}[5] = {} & S^2\mathcal{S} \otimes (\mathcal{S}' \oplus \mathcal{S}') \texttt{ generated by} \\ & \sigma^\alpha\sigma^\beta(\theta_L^2\lambda_L)_\gamma \texttt{ and } \sigma^\alpha\sigma^\beta(\theta_R^2\lambda_R)_\gamma \end{aligned} \tag{168}$$

$$\begin{aligned} E_1^{1,2}[5] = {} & \mathcal{S} \otimes V \otimes V \texttt{ generated by} \\ & \sigma^\alpha(\theta_L\lambda_L)^m(\theta_R\lambda_R)^n \end{aligned} \tag{169}$$

$$E_1^{4,0}[5] = E_1^{2,2}[5] \ = 0 \tag{170}$$

$$E_1^{1,3}[5] = E_1^{0,4}[5] \ = 0 \tag{171}$$

$$\begin{aligned} E_1^{3,1}[5] = {} & S^3\mathcal{S} \otimes (V \oplus V) \texttt{ generated by} \\ & \sigma^\alpha\sigma^\beta\sigma^\gamma(\lambda_L\theta_L)^m \texttt{ and } \sigma^\alpha\sigma^\beta\sigma^\gamma(\lambda_R\theta_R)^m \end{aligned} \tag{172}$$

$$E_1^{5,0}[5] = S^5\mathcal{S} \texttt{ generated by } \sigma^\alpha\sigma^\beta\sigma^\gamma\sigma^\delta\sigma^\epsilon \tag{173}$$

$$E_1^{4,1}[5] = E_1^{3,2}[5] \ = 0 \tag{174}$$

$$E_1^{2,3}[5] = E_1^{1,4}[5] = E_1^{0,5}[5] \ = 0 \tag{175}$$

5.6.2 $\lambda^2\theta^3$

Calculation of $E_2^{2,1}[5]$. The map $d_1 : E_2^{2,1}[5] \to E_2^{3,1}[5]$ is given by:

$$d_1\Big(\sigma^\alpha\sigma^\beta(\theta_L^2\lambda_L)_\gamma\Big) \simeq \sigma^\alpha\sigma^\beta(\Gamma^m\sigma)_\gamma(\theta_L\Gamma_m\lambda_L) \tag{176}$$

$$d_1\Big(\sigma^\alpha\sigma^\beta(\theta_R^2\lambda_R)_\gamma\Big) \simeq \sigma^\alpha\sigma^\beta(\Gamma^m\sigma)_\gamma(\theta_R\Gamma_m\lambda_R) \tag{177}$$

We observe that ker $(d_1 : E_1^{2,1}[5] \to E_1^{3,1}[5])$ is generated by the following classes:

$$(\lambda_L\Gamma^p\theta_L)(\theta_L\Gamma_p\Gamma_{mn}\sigma)\ \Gamma^{mn}\sigma + A(\lambda_L\Gamma^p\theta_L)(\theta_L\Gamma_p\sigma)\ \sigma \tag{178}$$

and the same class but with $\lambda_L \leftrightarrow \lambda_R$, where A is the coefficient such that[6]

$$(\sigma\Gamma_p\Gamma_{mn}\sigma)\ \Gamma^{mn}\sigma + A(\sigma\Gamma_p\sigma)\ \sigma = 0 \tag{179}$$

Since $E_1^{1,1}[5] = 0$, we conclude that $E_2^{2,1}[5] = S \oplus S$ generated by (178) and $\lambda_L \leftrightarrow \lambda_R$.

Calculation of $E_2^{0,2}[5]$ and $E_2^{1,2}[5]$. The kernel ker $d_1 : E_1^{0,2}[5] \to E_1^{1,2}[5]$ is generated by:

$$(\theta_L^3\lambda_L^2)_\alpha\ ,\quad (\theta_R^3\lambda_R^2)_\alpha \tag{180}$$

This means that $E_2^{0,2}[5]$ is generated by (180).

Also notice that im $d_1 : E_1^{0,2}[5] \to E_1^{1,2}[5]$ is generated by:

$$(\theta_R\Gamma^m\lambda_R)(\theta_L\Gamma^n\lambda_L)\Gamma_{mn}\sigma \tag{181}$$

$$\texttt{and}\quad (\theta_R\Gamma^m\lambda_R)(\theta_L\Gamma_m\lambda_L)\sigma \tag{182}$$

and that $d_1 : E_1^{1,2}[5] \to E_1^{2,2}[5]$ is zero because $E_1^{2,2}[5] = 0$. This means that $E_2^{1,2}[5]$ is generated by $(\theta_L\Gamma^m\lambda_L)(\theta_R\Gamma^n\lambda_R)\sigma^\alpha$ modulo (181) and (182).

[6]Notice that $(\sigma\Gamma_p\Gamma_{klmn}\sigma)\Gamma^{klmn}\sigma$ is linearly independent from $(\sigma\Gamma_p\sigma)\sigma$, therefore there is no such term in (178)

Calculation of $d_2 : E_2^{0,2}[5] \to E_2^{2,1}[5]$**.** It turns out that both $(\theta_L^3\lambda_L^2)_\alpha$ and $(\theta_R^3\lambda_R^2)_\alpha$ are acted upon nontrivially by d_2, and therefore do not survive in E_3. Notice that $d_2(\theta_L^3\lambda_L^2)$ is necessarily proportional to (178). This implies that

- $E_2^{0,2}[5]$ is cancelled against $E_2^{2,1}[5]$, *i.e.* $E_3^{0,2}[5] = E_3^{2,1}[5] = 0$

Vanishing of $E_3^{0,2}[5]$ implies that there is no cohomology of the type $\lambda^2\theta^3$. This can be demonstrated explicitly, in the following way. The leading term in λ_L would be $(\lambda_L\Gamma_m\theta)(\lambda_L\Gamma_n\theta)\Gamma_{mn}\theta$. But this is not annihilated by Q_R. We used a computer calculation (`L2T3Eqs` in Section 4.4.2) to confirm that indeed there is no cohomology of the type $\lambda^2\theta^3$.

Calculation of $E_3^{1,2}$ This is parallel to Section 5.5.3:

$$\begin{aligned} d\Big((\lambda_L\Gamma^m\theta_L)(\lambda_R\Gamma^n\theta_R)\sigma^\alpha\Big) &= (\lambda_L\Gamma^m\sigma)(\lambda_R\Gamma^n\theta_R)\sigma^\alpha + (\lambda_L\Gamma^m\theta_L)(\lambda_R\Gamma^n\sigma)\sigma^\alpha = \\ &= Q\Big((\theta_L\Gamma^m\sigma)(\lambda_R\Gamma^n\theta_R)\sigma^\alpha + (\theta_R\Gamma^n\sigma)(\lambda_L\Gamma^m\theta_L)\sigma^\alpha\Big) \end{aligned} \tag{183}$$

$$\begin{aligned} & d\Big((\theta_L\Gamma^m\sigma)(\lambda_R\Gamma^n\theta_R)\sigma^\alpha + (\theta_R\Gamma^n\sigma)(\lambda_L\Gamma^m\theta_L)\sigma^\alpha\Big) = \\ &= \sigma^\alpha(\sigma\Gamma^m\sigma)(\lambda_R\Gamma^n\theta_R) - \sigma^\alpha(\sigma\Gamma^n\sigma)(\lambda_L\Gamma^m\theta_L) - Q\Big(\sigma^\alpha(\sigma\Gamma^m\theta_L)(\theta_R\Gamma^n\sigma)\Big) \end{aligned} \tag{184}$$

By the same argument as in Section 5.5.3, the computation of the kernel of $d_2 : E_2^{1,2}[5] \to E_2^{3,1}[5]$ is equivalent to the problem of intertwining:

$$\sigma^\alpha\ \sigma^\gamma\ (a^m_{(L)}\Gamma_m\sigma)_\beta \ \ , \quad \sigma^\alpha\ \sigma^\gamma\ (a^m_{(R)}\Gamma_m\sigma)_\beta \tag{185}$$

$$\texttt{with}\ \ \sigma^\alpha\left((\sigma\Gamma^m\sigma)a^{(R)n} - (\sigma\Gamma^n\sigma)a^{(R)m}\right) \tag{186}$$

(where $a^n_{(L)}$ stands for $\lambda_L\Gamma^n\theta_L$ and $a^n_{(R)}$ for $\lambda_L\Gamma^n\theta_L$). This is only possible with one of the following options:

1. m and n in (186) are contracted
2. m and n in (186) are antisymmetrized
3. α and n in (186) are contracted through a Γ-matrix

But the first and the third options are not interesting, because the corresponding element of $E_2^{1,2}[5]$ is actually zero, *i.e.* in the image of $d_1 : E_1^{0,2}[5] \to E_1^{1,2}[5]$. Therefore we conclude that

- $E_3^{1,2}[5]$ is generated by:

$$(\lambda_L \Gamma^{[m}\theta_L)(\lambda_R \Gamma^{n]}\theta_R)\sigma^\alpha \tag{187}$$

  This corresponds to (33).

The following expression is an example of a nontrivial class:

$$\begin{aligned}&\lambda_{R+} \cup \theta_+ \cap \lambda_{L+} \cup \theta_+ \cap \lambda_{R+} + \frac{1}{4}||\lambda_{R+} \cup \lambda_{R+}||\theta_+ \cup \lambda_{L+} \cap \theta_+ - \\ &- \frac{1}{4}||\lambda_{R+} \cup \lambda_{R+}||(\theta_+ \cup \theta_+ \cap \lambda_{R+} + \lambda_{R+} \cup \theta_+ \cap \theta_+)\end{aligned} \tag{188}$$

We verified by a computer calculation (`L3T2Eqs` in Section 4.4.2) that this is not BRST-exact.

5.7 Case $N = 6$

$$E_1^{0,0}[6] = 0 \tag{189}$$

$$E_1^{1,0}[6] = E_1^{0,1}[6] = 0 \tag{190}$$

$$E_1^{2,0}[6] = E_1^{1,1}[6] = 0 \tag{191}$$

$$\begin{aligned} E_1^{0.2}[6] &= (\theta_L^2\lambda_L)_\alpha(\theta_R^2\lambda_R)_\beta\ , \\ &\quad (\theta_L^4\lambda_L^2)^m \texttt{ and } (\theta_R^4\lambda_R^2)^m \end{aligned} \tag{192}$$

$$E_1^{3,0}[6] = E_1^{2,1}[6] = E_1^{0,3}[6] = 0 \tag{193}$$

$$\begin{aligned} E_1^{1,2}[6] &= \sigma^\alpha(\theta_L\lambda_L)^m(\theta_R^2\lambda_R)_\beta \\ &\quad \sigma^\alpha(\theta_L^2\lambda_L)_\beta(\theta_R\lambda_R)^m \\ &\quad \sigma^\alpha(\theta_L^3\lambda_L^2)^\beta \texttt{ and } \sigma^\alpha(\theta_R^3\lambda_R^2)^\beta \end{aligned} \tag{194}$$

$$E_1^{4,0}[6] = E_1^{1,3}[6] = E_1^{0,4}[6] = 0 \tag{195}$$

$$E_1^{3,1}[6] = \sigma^\alpha\sigma^\beta\sigma^\gamma(\theta_L^2\lambda_L)_\delta \texttt{ and } \sigma^\alpha\sigma^\beta\sigma^\gamma(\theta_R^2\lambda_R)_\delta \tag{196}$$

$$E_1^{2,2}[6] = \sigma^\alpha\sigma^\beta(\theta_L\lambda_L)^m(\theta_R\lambda_R)^n \tag{197}$$

$$E_1^{5,0}[6] = E_1^{3,2}[6] = E_1^{2,3}[6] = 0 \tag{198}$$

$$E_1^{1,4}[6] = E_1^{0,5}[6] = 0 \tag{199}$$

$$\begin{aligned} E_1^{4,1}[6] &= \sigma^\alpha\sigma^\beta\sigma^\gamma\sigma^\delta(\theta_L\lambda_L)^m \texttt{ and} \\ &\quad \sigma^\alpha\sigma^\beta\sigma^\gamma\sigma^\delta(\theta_R\lambda_R)^m \end{aligned} \tag{200}$$

$$E_1^{6,0}[6] = \sigma^{\alpha_1}\cdots\sigma^{\alpha_6} \tag{201}$$

$$E_1^{5,1}[6] = E_1^{4,2} = E_1^{3,3} = 0 \tag{202}$$

$$E_1^{2,4}[6] = E_1^{1,5} = E_1^{0,6} = 0 \tag{203}$$

In fact $E_1^{0,2}[6]$ does not survive higher order corrections. Indeed, suppose that there is a nontrivial cohomology class of the form $\lambda^2\theta^4$. It should have the form:

$$\lambda_L^2\theta^4 + \ldots \tag{204}$$

where $\ldots$ stands for terms containing $\lambda_L\lambda_R\theta^4$ and $\lambda_R^2\theta^4$. The derivative of such an expression with respect to θ would have started with $\lambda_L^2\theta^3$ representing a nontrivial cohomology class of the type $\lambda^2\theta^3$. But there is no such class — see Section 5.6. We have confirmed by a symbolic computation that there is no such class (`L2T4Eqs` in Section 4.4.2).

Now consider a class of the order $\lambda^3\theta^3$. Then the leading term in λ_L should be a linear combination of the following two:

$$\theta_+^\alpha \cap \lambda_{L+} \cup \theta_+ \cap \theta_+ \cup \lambda_{L+} \cap \lambda_{R+}^\beta \tag{205}$$

$$\texttt{and} \quad \lambda_{R+}^\alpha \cap \lambda_{L+} \cup \theta_+ \cap \theta_+ \cup \lambda_{L+} \cap \theta_+^\beta \tag{206}$$

Both can be made $Q^{[0]}$-closed by adding the terms of the type $\lambda_{R+}^2\lambda_{L+}\theta^3$ and $\lambda_{R+}^3\theta^3$. The part antisymmetric in $\alpha \leftrightarrow \beta$ then becomes $Q^{[0]}$-exact:

$$\begin{aligned}
&- \theta_+^\alpha \cap \lambda_{L+} \cup \theta_+ \cap \theta_+ \cup \lambda_{L+} \cap \lambda_{R+}^\beta \; + \; \lambda_{R+}^\alpha \cap \lambda_{L+} \cup \theta_+ \cap \theta_+ \cup \lambda_{L+} \cap \theta_+^\beta \; = \\
&= \left(\lambda_{L+}\frac{\partial}{\partial\theta_+} + \lambda_{R+}\frac{\partial}{\partial\theta_+}\right)\Big(\frac{5}{8}\,\theta_+^\alpha \cap \lambda_{L+} \cup \theta_+ \cap \theta_+ \cup \lambda_{L+} \cap \theta_+^\beta + \\
&\qquad + \texttt{some}\ [\lambda_L\lambda_R\theta^4] + \texttt{some}\ [\lambda_R^2\theta^4]\Big)
\end{aligned} \tag{207}$$

But the part symmetric in $(\alpha \leftrightarrow \beta)$ does represent a nontrivial cohomology class. There must be also a class of the type (58) and (59), but we have not verified it.

The general theory behind the classed of the order $\lambda^3\theta^3$ is the following. We explained in Section 5.6 that d_1 acts nontrivially on $(\theta_L^2\lambda_L)(\theta_R\lambda_R)$ and $(\theta_R^2\lambda_R)(\theta_L\lambda_L)$. This was the reason why they do not survive in $E_2[5]$. However, let us now consider the following $\mathbf{Z}_2^{LR}$-odd element of $E_1^{1,2}[6]$:

$$(\sigma\Gamma_{abc}\Gamma_n\Gamma_m\theta_L)(\theta_L\Gamma^m\lambda_L)(\theta_R\Gamma^n\lambda_R) + (L \leftrightarrow R) \tag{208}$$

The d_1 annihilates this element. Indeed, the d_1 of this element is necessarily proportional to:

$$(\sigma\Gamma_{abc}\Gamma_{mn}\sigma)(\theta_L\Gamma^m\lambda_L)(\theta_R\Gamma^n\lambda_R) \tag{209}$$

which is $\mathbf{Z}_2^{LR}$-even; therefore d_1 of (208) is zero.

The d_1 of (208) being zero implies that the d_2 of it is well defined. The value of d_2 on (208) is of the form:

$$[\sigma^3\lambda_L\theta_L^2] + [\sigma^3\lambda_R\theta_R^2] \tag{210}$$

On the other hand, let us consider another $\mathbf{Z}_2^{LR}$-odd element, of the form:

$$(\sigma\Gamma_{abc}\Gamma_n\Gamma_m\theta_L)(\theta_L\Gamma^m\lambda_L)(\theta_L\Gamma^n\lambda_L) + (L \leftrightarrow R) \tag{211}$$

The d_2 of (211) is of the same form as the d_2 of (211), namely of the form (210). Therefore:

- some linear combination of (208) and (211) survives on $E_\infty^{1,2}[6]$

This gives (34).

5.8 $N = 7$

There are no nontrivial classes of the form $\lambda^2\theta^5$. To prove this, we have to consider the term with the highest number of λ_L. It has to be annihilated by $\lambda_L \frac{\partial}{\partial\theta}$. Because the cohomology of $\lambda_L \frac{\partial}{\partial\theta}$ in degrees $\lambda_L^2\theta^5$, $\lambda_L\theta^5$ and $\lambda_L^0\theta^5$ is zero, this can be always gauged away.

However, there is a nontrivial class of the form $\lambda^3\theta^4$ — see (60) or (35).

5.9 $N = 8$

At the level eight we have a nontrivial $E_1^{0,3}[8]$ generated by $\lambda_L^3\theta_L^5$, $\lambda_R^3\theta_R^5$, $(\lambda_L^2\theta_L^4)(\lambda_R\theta_R)$, $(\lambda_R^2\theta_R^4)(\lambda_L\theta_L)$, $(\lambda_L^2\theta_L^3)(\lambda_R\theta_R^2)$, $(\lambda_R^2\theta_R^3)(\lambda_L\theta_L^2)$. The classes surviving on E_2 (*i.e.* annihilated by the d_1) are:

- $\lambda_L^3\theta_L^5$ and $\lambda_R^3\theta_R^5$
- some linear combinations of the type:

$$a(\lambda_L^2\theta_L^4)(\lambda_R\theta_R) + b(\lambda_L^2\theta_L^3)(\lambda_R\theta_R^2) \quad \texttt{and} \quad a(\lambda_R^2\theta_R^4)(\lambda_L\theta_L) + b(\lambda_R^2\theta_R^3)(\lambda_L\theta_L^2) \tag{212}$$

What happens when we pass to E_3? Consider $d_2 : E_2^{0,3}[8] \to E_2^{2,2}[8]$. The potential obstacle is in $E_2^{2,2}[8]$.

Let us look at $E_1^{2,2}[8]$:

$$E_1^{2,2}[8] \ : \sigma^\alpha\sigma^\beta(\lambda_L^2\theta_L^4)^m \ \texttt{and} \ \sigma^\alpha\sigma^\beta(\lambda_R^2\theta_R^4)^m \ \texttt{and} \ \sigma^\alpha\sigma^\beta(\lambda_L\theta_L^2)_\gamma(\lambda_R\theta_R^2)_\delta \tag{213}$$

Also notice that $E_1^{1,2}[8] = 0$, and therefore nothing in $E_1^{2,2}[8]$ is in the image of d_1. We will now use the symmetry $\mathbf{Z}_2^{LR}$ which was described in Section 4.2. Let us look at those elements of $E_1^{2,2}[8]$ which are scalars and are of the type $[\sigma^2\lambda_L\lambda_R\theta_L^2\theta_R^2]$. They are coming from $\sigma^\alpha\sigma^\beta(\lambda_L\theta_L^2)_\gamma(\lambda_R\theta_R^2)_\delta$. They are all even[7] under $\mathbf{Z}_2^{LR}$. We therefore avoid this obstacle, if we simply restrict ourselves to $\mathbf{Z}_2^{LR}$-odd elements.

It is enough to get rid of the obstacles $\sigma^\alpha\sigma^\beta(\lambda_L^2\theta_L^4)^m$ and $\sigma^\alpha\sigma^\beta(\lambda_R^2\theta_R^4)^m$. Let us consider the $\mathbf{Z}_2^{LR}$-odd combination of the form:

$$\begin{aligned} &\Big((\lambda_L^3\theta_L^5) + a(\lambda_L^2\theta_L^4)^m(\lambda_R\theta_R)^m + b(\lambda_L^2\theta_L^3)^\alpha(\lambda_R\theta_R^2)_\alpha\Big) - \\ &- \Big((\lambda_R^3\theta_R^5) + a(\lambda_R^2\theta_R^4)^m(\lambda_L\theta_L)^m + b(\lambda_R^2\theta_R^3)^\alpha(\lambda_L\theta_L^2)_\alpha\Big) \end{aligned} \tag{214}$$

[7] Indeed, the odd elements would be those containing $\lambda_L\theta_L^2$ and $\lambda_R\theta_R^2$ through $((\lambda_L\theta_L^2)\Gamma_{pqr}(\lambda_R\theta_R^2))$. But $\sigma^\alpha\sigma^\beta$ is symmetric in $\alpha \leftrightarrow \beta$ and therefore does not contain a three-form.

The ratio of the coefficients a and b is fixed as in (212), in other words:

$$d_1\Big((\lambda_L^3\theta_L^5) + a(\lambda_L^2\theta_L^4)^m(\lambda_R\theta_R)^m + b(\lambda_L^2\theta_L^3)^\alpha(\lambda_R\theta_R^2)_\alpha\Big) \;=\; 0 \tag{215}$$

Now let us adjust a and b (keeping their ratio) so that:

$$\begin{aligned} & d_2\Big((\lambda_L^3\theta_L^5) + a(\lambda_L^2\theta_L^4)^m(\lambda_R\theta_R)^m + b(\lambda_L^2\theta_L^3)^\alpha(\lambda_R\theta_R^2)_\alpha\Big) \;= \\ & = c_1(\sigma\Gamma^m\sigma)\left((\lambda_L\theta_L^2)\Gamma^m(\lambda_R\theta_R^2)\right) + c_5(\sigma\Gamma^{m_1\ldots m_5}\sigma)\left((\lambda_L\theta_L^2)\Gamma^{m_1\ldots m_5}(\lambda_R\theta_R^2)\right) \end{aligned} \tag{216}$$

with some c_1 and c_2; the point is that it is possible to choose a and b so that the $d_2(\ldots)$ does not have any terms proportional to $\sigma\sigma(\lambda_L^2\theta_L^4)$. (Notice that the absence of the terms proportional to $\sigma\sigma(\lambda_R^2\theta_R^4)$ is automatic due to the $\mathbf{Z}_2^{LR}$-symmetry.) This means that we removed the obstacles $\sigma^\alpha\sigma^\beta(\lambda_L^2\theta_L^4)^m$ and $\sigma^\alpha\sigma^\beta(\lambda_R^2\theta_R^4)^m$.

$$\begin{aligned} d_2\;\Big[\; & \Big((\lambda_L^3\theta_L^5) + a(\lambda_L^2\theta_L^4)^m(\lambda_R\theta_R)^m + b(\lambda_L^2\theta_L^3)^\alpha(\lambda_R\theta_R^2)_\alpha\Big) - \\ & - \Big((\lambda_R^3\theta_R^5) + a(\lambda_R^2\theta_R^4)^m(\lambda_L\theta_L)^m + b(\lambda_R^2\theta_R^3)^\alpha(\lambda_L\theta_L^2)_\alpha\Big)\;\Big] = 0 \end{aligned} \tag{217}$$

Furthermore:

$$d_3 \;:\; E_3^{0,3}[8] \to E_3^{3,1}[8] = 0 \tag{218}$$

$$d_4 \;:\; E_4^{0,3}[8] \to E_4^{4,0}[8] = 0 \tag{219}$$

We conclude that with this special choice of a and b (214) actually survives all the way up to E_∞. This means that there must be a representative function of $\theta_L + \theta_R$; this is Eq. (36).

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